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# General Connection Formulae for Liouville-Green Approximations in the Complex Plane

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# GENERAL CONNECTION FORMULAE FOR LIOUVILLE–GREEN APPROXIMATIONS IN THE COMPLEX PLANE

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## CONTENTS



This paper is concerned with differential equations of the form

$$d^2w/dz^2 = \{u^2 f(u,z) + g(u,z)\}\, w,$$

in which $u$ is a positive parameter and $z$ is a complex variable ranging over a simply connected open domain $\boldsymbol{D}$ that is not necessarily one-sheeted, and may be bounded or unbounded.

In the first part we assume that for each value of $u$ the function $(z-c)^{2-m} f(u,z)$ is holomorphic and non-vanishing throughout $\boldsymbol{D}$, where $c$ is an interior point of $\boldsymbol{D}$ and $m$ is a positive constant. It is also assumed that $g(u,z)$ is holomorphic in $\boldsymbol{D}$, punctured at $c$,

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and $g(u, z) = O\{(z-c)^{\gamma-1}\}$ as $z \to c$, where $\gamma$ is another positive constant. Thus $c$ is a fractional transition point of the differential equation of multiplicity (or order) $m-2$, and there are no other transition points in $\boldsymbol{D}$. Uniform asymptotic approximations for the solutions, when $u$ is large, are constructed in terms of Bessel functions of order $1/m$, complete with error bounds.

In the second part the Bessel function approximants are replaced by their uniform asymptotic approximations for large argument, yielding the connection formulae for the Liouville–Green (or J.W.K.B.) approximations to the solutions, again complete with error bounds. These results are then applied to solve the general problem of connecting the Liouville–Green approximations when $\boldsymbol{D}$ contains any (finite) number of transition points of arbitrary multiplicities, integral or fractional.

The third, and concluding, part illustrates the theory by means of three examples.

An appendix describes a numerical method for the automatic computation and plotting of the boundary curves of the Liouville–Green approximations, defined by

$$\mathrm{Re} \int_c^z f^{\frac{1}{2}}(u, t)\, \mathrm{d}t = 0,$$

where $c$ again denotes a transition point.

## 1. Introduction and summary

### 1.1. *Introduction*

The problem of constructing asymptotic solutions of differential equations of the form

$$\mathrm{d}^2 w/\mathrm{d}x^2 = \{u^2 f(u, x) + g(u, x)\} w \tag{1.1}$$

for large values of the positive real parameter $u$ has been studied by many physicists and mathematicians, including the present writer. In Olver (1977*a*) I considered the case in which the independent variable $x$ ranges over a finite or infinite open interval, within which $f(u, x)$ and $g(u, x)$ are free from singularity and $f(u, x)$ has a single zero of arbitrary multiplicity. Zeros of $f(u, x)$ are called *turning points* or *transition points* of (1.1), and in Olver (1977*b*) I considered the necessary modifications when the interval of integration is permitted to contain any (finite) number of turning points of arbitrary multiplicities. The object in these two papers was to connect the Liouville–Green approximations† that represent the solutions in the neighbourhood of an end-point of the interval of integration with the corresponding approximations valid near the other end-point.

The present paper develops a similar theory when the independent variable, which we now denote by $z$, ranges over a bounded or unbounded domain $\boldsymbol{D}$ in the complex plane. This extension has a variety of promising physical applications, including scattering and optical potential problems in quantum mechanics (Pokrovskii & Khalatnikov 1961; Mott & Massey 1965; Brander 1966; Berry & Mount 1972; Child 1974; Connor, Jakubetz & Sukumar 1976; Knoll & Schaeffer 1976), transmission of radio waves (Budden & Smith 1974), trapping of water waves (Lozano & Meyer 1976), and hydrodynamic instability (Drazin 1974).

A new feature in the complex case is that the functions $f(u, z)$ and $g(u, z)$ in the differential equation

$$\mathrm{d}^2 w/\mathrm{d}z^2 = \{u^2 f(u, z) + g(u, z)\} w \tag{1.2}$$

are required to be analytic functions of the complex variable $z$. Another important difference is that the Liouville–Green (L.G.) approximations are valid only in restricted regions of $\boldsymbol{D}$. Suppose,

† Also known as J.W.K.B. approximations.



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for example, that $\boldsymbol{D}$ contains a single turning point $c$ of multiplicity $m-2$, where $m$ is an integer not less than 2. Then we first construct the curves

$$\operatorname{Re}\int_c^z f^{\frac{1}{2}}(u,t)\,\mathrm{d}t = 0,$$

called the *principal curves* (or *anti-Stokes lines*) *associated with* $c$. It is easily verified that $m$ such curves emerge from $c$, dividing $\boldsymbol{D}$ into $m$ regions, which we call the *principal regions*† *associated with* $c$. Each L.G. approximation for a given solution of (1.2) is valid in at most three adjoining principal regions, and then only after deletion of a domain that contains the boundary of their union.

Hitherto, methods used for connecting L.G. approximations in the complex plane have depended on the construction of *transition* (or *transmission*) *matrices* that relate pairs of solutions in two adjoining principal regions. To pass through several regions we form the product of the corresponding transition matrices. In Olver (1977*a*) methods of this type were classified as *lateral* or *pseudo-lateral*, depending on the way in which the transition matrices are derived. The earliest investigation of transition matrices for turning points of arbitrary multiplicity appears to be the formal analysis of Heading (1962, pp. 89–93, and 110–115). Leung (1975) supplied a rigorous proof, based on a uniform reduction theorem due to Sibuya (1974), for the case in which $f(u,z)$ is a polynomial in $z$ that is independent of $u$, $g(u,z) \equiv 0$ and $\boldsymbol{D}$ contains a single multiple turning point. More recently, Leung (1977) has applied his results to solve an eigenvalue problem involving several turning points. Further historical details may be found in Olver (1977*a*, §6) and Olver (1977*b*, §6) and will not be repeated here.

In the present paper it is shown how to continue a solution that is recessive in any given principal region directly to *any* other principal region associated with the same turning point, without the need for passage through each of the intervening regions. The method is of *central connection* type, as defined by Wasow (1968) and Olver (1977*a*). The procedure has the obvious advantage of reducing substantially the number of steps needed in the connection process. When $\boldsymbol{D}$ contains $n$ turning points, at most $n$ steps are needed to trace any solution, whatever the multiplicities of the turning points. In practice, the number of steps is often very much less than $n$. Also, in contrast to the procedures of Heading and Leung, there are no restrictions on the configuration of the turning points.‡

The present theory differs from earlier investigations in two other respects. First, explicit and realistic error bounds are found for the approximate coefficients in the connection formulae in a form that is suitable for a single transition point or several transition points. In the real-variable theory of Olver (1977*a*, *b*) error bounds were constructed only for the case of a single transition point. The present extension to several transition points was stimulated by the work of Taylor (1978).

Secondly, the theory admits *fractional transition points*, that is, points $c$ at which $f(u,z)/(z-c)^{m-2}$ is analytic, $m$ now denoting *any* positive constant. In general, such points are branch-points of the solutions of the differential equation. The analysis in these cases is more difficult, but apart from the need to introduce Riemann sheets the final connection formulae are essentially the same. This extension covers the commonly occurring case in which $f(u,z)$ has a simple pole at $c$; in this event $m = 1$.

† Other names are *Stokes regions* (Wasow 1968) and *principal subdomains* (Olver 1965).

‡ *Added in proof*, 8 *May* 1978. Restrictions on the number and configuration of the turning points have also been removed in a recent paper by Heading (1977). The object in this reference is to develop a method that can be applied easily, and it is achieved by use of pseudo-lateral connection with formal analysis.

### 1.2. *Summary*

The paper is divided into three parts, the contents of which are as follows.

In part A, comprising §§2–4, the given complex domain $\boldsymbol{D}$ is assumed to contain a single transition point of any multiplicity, integral or fractional. In §2 we collect relevant properties of standard solutions of the basic comparison equation

$$d^2w/dt^2 = \tfrac{1}{4}m^2t^{m-2}w, \tag{1.3}$$

especially connection formulae and uniform asymptotic approximations for large $|t|$. We also introduce auxiliary weight, modulus and phase functions. In §3 we state and prove the main approximation theorem for solutions of (1.2). This theorem furnishes approximations for large values of the parameter $u$ in terms of solutions of (1.3). Except for the usual possible shadow zones (see, for example, Cherry 1950; Olver 1974, pp. 223 and 417) these approximations are uniformly valid in the whole of $\boldsymbol{D}$, even when $\boldsymbol{D}$ is unbounded or has irregular singularities of (1.2) on its boundary. This problem is of considerable importance in its own right, quite apart from the connection formula problem. It has been considered previously by Langer (1932, 1935) and Olver (1977*c*), but the asymptotic solutions constructed in these references are not the ones that are pertinent to the present investigation, and new proofs are needed, especially when $m$ is not an integer. The approximations found in §3 are accompanied by inequalities satisfied by the error terms, and are valid when the coefficients $f(u,z)$ and $g(u,z)$ in (1.2) depend on the parameter $u$ in a fairly general manner. For the approximations to be meaningful, however, the error terms must vanish as $u \to \infty$, and in §4 we derive asymptotic estimates of the error bounds in the common case in which $f(u,z)$ and $g(u,z)$ are independent of $u$.

In part B, comprising §§5–7, the uniform approximations of §3 are re-approximated away from the transition point by replacing the basic functions, that is, the standard solutions of (1.3), by their asymptotic approximations for large argument. This yields the L.G. approximations in each principal region. The main connection theorem, giving the L.G. approximations complete with inequalities satisfied by the error terms, is stated and proved in §5. In §6 we derive asymptotic estimates of the error bounds in the case when $f(u, z)$ and $g(u, z)$ are independent of $u$ by application of the results of §4. Applications of the theorems of §§5 and 6 are discussed in §7. To solve the general problem in which there are any number of transition points, it suffices to consider a domain $\boldsymbol{D}$ containing two transition points of arbitrary multiplicities. Two distinct cases arise, depending whether or not these points are joined by a common principal curve.

Part C, comprising §§8–10, illustrates the theory by means of three examples. The first (§8) is an eigenvalue problem involving four real turning points of multiplicities 2, 1, 4 and 3. This example was solved previously in Olver (1977*b*, §5) and therefore serves as a check on the present analysis. The second example (§9) has three triple turning points, distributed equidistantly around the unit circle. The final example (§10) involves four singularities. Two of these are located on the real axis and are simple poles of $f(u, z)$; thus $m = 1$ for each. The other two are branch-points of $f(u,z)$ situated on the imaginary axis, and $m = \frac{5}{2}$ for each.

An appendix outlines the numerical method that was used for the automatic computation and plotting of the principal curves for the examples described in part C.

For the reader who is concerned only with applying the connection formulae in the case when the functions $f(u,z)$ and $g(u,z)$ are independent of $u$, and not with the proofs of the theorems or the evaluation of error bounds, the relevant parts of this paper are §§5.1, 6.1, 6.2, 6.4, 7.1–7.4, the examples treated in §§8–10, and the appendix.

# PART A. UNIFORM APPROXIMATIONS FOR THE SOLUTIONS

## 2. Standard solutions of the basic equation

### 2.1. *Primary solution*

In this section we derive properties of solutions of the equation

$$d^2w/dt^2 = \tfrac{1}{4}m^2t^{m-2}w. \tag{2.1}$$

For the case in which $t$ is a real variable and $m-2$ is a non-negative integer, Olver (1977*a*), with a slight change in notation, adopted the function

$$U(t) = (2t/\pi)^{\frac{1}{2}}K_{1/m}(t^{\frac{1}{2}m}) \tag{2.2}$$

as a standard solution, $K$ denoting the modified Bessel function in the usual notation. We continue to employ this solution in the more general circumstances which we now contemplate, that is, when $t$ is a complex variable and $m$ has any positive real value. The functions on the right hand side of (2.2) are understood to assume their principal values when $\operatorname{ph} t = 0$, and be defined by continuity for other values of $\operatorname{ph} t$.

Properties of $U(t)$ are easily deduced from those of Bessel functions, given for example in National Bureau of Standards (1964, ch. 9). Thus if $m$ is an integer such that $m \geqslant 2$, then $U(t)$ is entire. For other values of $m$, $U(t)$ has a branch-point at $t = 0$ and is analytic at all other finite points of the $t$-plane.

As $t \to 0$

$$U(t) \to \frac{2^{(2-m)/(2m)}}{\pi^{\frac{1}{2}}}\Gamma\left(\frac{1}{m}\right). \tag{2.3}$$

In the same circumstances the asymptotic form of $U'(t)$ is given by

$$-\frac{2^{(2-5m)/(2m)}m}{\pi^{\frac{1}{2}}}\Gamma\left(\frac{1}{m}-1\right)t^{m-1}, \quad \frac{\ln t}{(8\pi)^{\frac{1}{2}}} \quad \text{or} \quad \frac{2^{-(2+m)/(2m)}}{\pi^{\frac{1}{2}}}\Gamma\left(-\frac{1}{m}\right), \tag{2.4}$$

according as $m < 1$, $m = 1$ or $m > 1$.

As $t \to \infty$

$$U(t) \sim t^{\frac{1}{4}(2-m)}\exp(-t^{\frac{1}{2}m}), \quad U'(t) \sim -\tfrac{1}{2}mt^{\frac{1}{4}(m-2)}\exp(-t^{\frac{1}{2}m}), \tag{2.5}$$

each of these relations being valid when $|\operatorname{ph} t| \leqslant (3-\delta)\pi/m$. Here $\delta$ denotes any constant such that $0 < \delta < 1$.

We shall use the approximations (2.5) only in the sector $|\operatorname{ph} t| \leqslant 2\pi/m$, and in this region uniform error bounds are supplied by the following results:

$$U(t) = t^{\frac{1}{4}(2-m)}\exp(-t^{\frac{1}{2}m})\{1+\vartheta(t)\}, \tag{2.6}$$

$$d\{t^{\frac{1}{4}(m-2)}U(t)\}/dt = -\tfrac{1}{2}mt^{\frac{1}{2}(m-2)}\exp(-t^{\frac{1}{2}m})\{1+\vartheta^1(t)\}, \tag{2.7}$$

where

$$|\vartheta(t)|, |\vartheta^1(t)| \leqslant \Theta(t^{\frac{1}{2}m}) \quad (|\operatorname{ph} t| \leqslant 2\pi/m), \tag{2.8}$$

and

$$\Theta(t) = \exp\left|\frac{\pi(4-m^2)}{8m^2t}\right| - 1. \tag{2.9}$$

Thus $\vartheta(t)$ and $\vartheta^1(t)$ are both $O(t^{-\frac{1}{2}m})$ as $t \to \infty$ in the sector. To verify (2.8) set $z = t^{\frac{1}{2}m}$ and

$\vartheta(t) = \mathrm{e}^z h(z)$. Then from (2.6) and (2.7) we find that $\vartheta^1(t) = -\mathrm{e}^z h'(z)$. Using the differential equation

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}\{z^{\frac{1}{2}}K_{1/m}(z)\} = \left(1+\frac{4-m^2}{4m^2z^2}\right)\{z^{\frac{1}{2}}K_{1/m}(z)\},$$

and making the substitution

$$z^{\frac{1}{2}}K_{1/m}(z) = (\tfrac{1}{2}\pi)^{\frac{1}{2}}\{\mathrm{e}^{-z}+h(z)\},$$

we obtain
$$h''(z)-h(z) = (4-m^2)\,(4m^2z^2)^{-1}\{\mathrm{e}^{-z}+h(z)\},$$

and hence, by variation of parameters,

$$h(z) = \frac{4-m^2}{8m^2}\int_z^\infty (\mathrm{e}^{v-z}-\mathrm{e}^{z-v})\,\{\mathrm{e}^{-v}+h(v)\}\frac{\mathrm{d}v}{v^2}.$$

On solving this integral equation by means of theorem 10.2 of Olver (1974, ch. 6), and using the relations (13.03) and (13.05) of the same reference, we conclude that $|h(z)|$ and $|h'(z)|$ are both bounded by $|\mathrm{e}^{-z}|\,\Theta(z)$ when $|\mathrm{ph}\,z| \leqslant \pi$. This result is clearly equivalent to (2.8). Compare also Olver (1964, §§5 and 7).

Asymptotic forms of $U(t)$ as $t\to\infty$ in other phase ranges will be found in §2.4 below.

When $m-2$ is a positive integer further properties of $U(t)$, particularly the location of its zeros, may be derived from results of Swanson & Headley (1967) and Headley & Barwell (1975). In the notation of these authors

$$U(t) = (\tfrac{1}{2}\pi)^{\frac{1}{2}}(2/m)^{1/m}m\,\mathrm{cosec}\,(\pi/m)\,A_{m-2}\{(\tfrac{1}{2}m)^{2/m}t\}.$$

### 2.2. *Secondary solutions*

We define†

$$U_j(t) = U(t\,\mathrm{e}^{-2j\pi\mathrm{i}/m}) \quad (j=0,\ \pm1,\ \pm2,\ldots). \tag{2.10}$$

Clearly $U_j(t)$ is a solution of equation (2.1) that is recessive as $t\to\infty$ in the sector $\boldsymbol{S}_j$, defined by

$$(2j-1)\,\pi/m \leqslant \mathrm{ph}\,t \leqslant (2j+1)\,\pi/m. \tag{2.11}$$

These sectors are indicated in figure 1. From (2.6) to (2.9) we obtain

$$U_j(t) = \mathrm{i}^j\mathrm{e}^{-j\pi\mathrm{i}/m}t^{\frac{1}{4}(2-m)}\exp\{-(t\,\mathrm{e}^{-2j\pi\mathrm{i}/m})^{\frac{1}{2}m}\}\{1+\vartheta_j(t)\}, \tag{2.12}$$

$$\mathrm{d}\{t^{\frac{1}{4}(m-2)}U_j(t)\}/\mathrm{d}t = -\tfrac{1}{2}m\mathrm{i}^{-j}\mathrm{e}^{-j\pi\mathrm{i}/m}t^{\frac{1}{2}(m-2)}\exp\{-(t\,\mathrm{e}^{-2j\pi\mathrm{i}/m})^{\frac{1}{2}m}\}\{1+\vartheta_j^1(t)\}, \tag{2.13}$$

where
$$|\vartheta_j(t)|,\ |\vartheta_j^1(t)| \leqslant \Theta(t^{\frac{1}{2}m}), \tag{2.14}$$

valid when
$$(2j-2)\,\pi/m \leqslant \mathrm{ph}\,t \leqslant (2j+2)\,\pi/m. \tag{2.15}$$

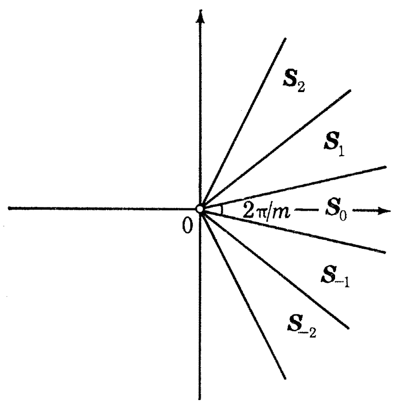


FIGURE 1. Sectors $\boldsymbol{S}_j$.

† This notation should not be confused with that of Olver (1977*a*).

When $m > 1$ the Wronskian of any pair of solutions $U_j(t)$ and $U_k(t)$ is easily calculated by letting $t \to 0$ and referring to (2.3) and (2.4); thus

$$\mathscr{W}\{U_j(t), U_k(t)\} = (e^{-2k\pi i/m} - e^{-2j\pi i/m})\, U(0)\, U'(0) = mi\, e^{-(j+k)\pi i/m} \lambda_{j,k}, \tag{2.16}$$

where

$$\lambda_{j,k} = \frac{\sin(k-j)\pi/m}{\sin \pi/m}. \tag{2.17}$$

The validity of (2.16) when $0 < m \leqslant 1$ is then inferred by temporarily supposing $m$ to be a complex variable and appealing to analytic continuation. When $1/m$ happens to be an integer the ratio of the sine functions in (2.17) is to be replaced by its limiting value, that is,

$$\lambda_{j,k} = (-)^{(k-j-1)/m}(k-j), \quad 1/m = \text{an integer}. \tag{2.18}$$

From (2.16), (2.17) and (2.18) we conclude that $U_j(t)$ and $U_k(t)$ are linearly independent solutions of (2.1), provided that

$$k \neq j, \quad \text{when } 1/m = \text{an integer}, \tag{2.19}$$

or

$$k \neq j \pmod m, \quad \text{when} \quad 1/m \neq \text{an integer}. \tag{2.20}$$

We also note that if $1/m$ is not an integer and $k = j \pmod m$, then $m$ is necessarily rational and the sector $\boldsymbol{S}_k$ is a superimposition of $\boldsymbol{S}_j$ in the complex plane. In particular, if $m$ is an integer such that $m \geqslant 2$, then there are only $m$ distinct sectors $\boldsymbol{S}_j$ and $m$ distinct solutions $U_j(t)$.

### 2.3. *Connection formulae*

The linear relation satisfied by any three secondary solutions may be found as follows. From the connection formulae for the modified Bessel functions given on page 376 of National Bureau of Standards (1964) we have

$$\sin(\pi/m)\, K_{1/m}(z\, e^{-j\pi i}) = \sin(\pi/m)\, e^{j\pi i/m} K_{1/m}(z) + \pi i \sin(j\pi/m)\, I_{1/m}(z).$$

On replacing $j$ by $k$ and $l$ in turn and eliminating $K_{1/m}(z)$ and $I_{1/m}(z)$ from the resulting three equations, we arrive at

$$\sin\{(k-l)\pi/m\} K_{1/m}(z\, e^{-j\pi i}) + \sin\{(l-j)\pi/m\} K_{1/m}(z\, e^{-k\pi i}) + \sin\{(j-k)\pi/m\} K_{1/m}(z\, e^{-l\pi i}) = 0.$$

Hence from (2.2) and (2.10) the desired formula is seen to be

$$\sin\{(k-l)\pi/m\}\, e^{j\pi i/m} U_j(t) + \sin\{(l-j)\pi/m\}\, e^{k\pi i/m} U_k(t) + \sin\{(j-k)\pi/m\}\, e^{l\pi i/m} U_l(t) = 0. \tag{2.21}$$

Special cases that will be required in the next subsection are given by

$$U_j(t) = \mp e^{(k-j\pm 1)\pi i/m} \lambda_{j,k}\, U_{k\pm 1}(t) \pm e^{(k-j)\pi i/m} \lambda_{j,k\pm 1}\, U_k(t), \tag{2.22}$$

the upper or lower signs being taken consistently throughout.

### 2.4. *Further asymptotic forms of the secondary solutions*

Let us assume that $2k\pi/m \leqslant \operatorname{ph} t \leqslant (2k+2)\pi/m$. Then $\operatorname{ph} t$ satisfies (2.15) with $j$ replaced by either $k$ or $k+1$. Substituting in the right hand side of (2.22), with upper signs, by means of (2.12) we derive

$$\begin{aligned} U_j(t) = {} & i^{k-1} e^{-j\pi i/m} \lambda_{j,k}\, t^{\frac{1}{4}(2-m)} \exp\{(t\, e^{-2k\pi i/m})^{\frac{1}{2}m}\}\{1 + \vartheta_{k+1}(t)\} \\ & + i^k e^{-j\pi i/m} \lambda_{j,k+1}\, t^{\frac{1}{4}(2-m)} \exp\{-(t\, e^{-2k\pi i/m})^{\frac{1}{2}m}\}\{1 + \vartheta_k(t)\}, \end{aligned} \tag{2.23}$$

where $\vartheta_k(t)$ and $\vartheta_{k+1}(t)$ are subject to (2.14).

At this stage it is convenient to introduce some terminology apropos the sector $\boldsymbol{S}_k$. First, we call the ray $\mathrm{ph}\, t = (2k+1)\,\pi/m$ the *left boundary of* $\boldsymbol{S}_k$, owing to its orientation when viewed from the vertex at $t = 0$. Similarly $\mathrm{ph}\, t = (2k-1)\,\pi/m$ is called the *right boundary of* $\boldsymbol{S}_k$. Next, we call the sectors

$$2k\pi/m \leqslant \mathrm{ph}\, t \leqslant (2k+1)\,\pi/m, \quad (2k-1)\,\pi/m \leqslant \mathrm{ph}\, t \leqslant 2k\pi/m,$$

the *left* and *right parts of* $\boldsymbol{S}_k$, respectively. And if $\delta$ is a constant such that $0 < \delta < 1$ we call

$$(2k-\delta)\,\pi/m \leqslant \mathrm{ph}\, t \leqslant (2k+\delta)\,\pi/m$$

an *internal part of* $\boldsymbol{S}_k$.

With these definitions, (2.23) is valid in the left part of $\boldsymbol{S}_k$, and also in the right part of $\boldsymbol{S}_{k+1}$. The corresponding asymptotic approximation in the right part of $\boldsymbol{S}_k$ and the left part of $\boldsymbol{S}_{k-1}$ may be obtained by use of (2.22), with lower signs, or by changing $k$ into $k-1$ in (2.23). Either way yields

$$U_j(t) = \mathrm{i}^{k-1}\mathrm{e}^{-j\pi\mathrm{i}/m}\lambda_{j,k}\, t^{\frac{1}{4}(2-m)} \exp\{(t\,\mathrm{e}^{-2k\pi\mathrm{i}/m})^{\frac{1}{2}m}\}\{1+\vartheta_{k-1}(t)\}$$
$$- \mathrm{i}^{k}\mathrm{e}^{-j\pi\mathrm{i}/m}\lambda_{j,k-1}\, t^{\frac{1}{4}(2-m)} \exp\{-(t\,\mathrm{e}^{-2k\pi\mathrm{i}/m})^{\frac{1}{2}m}\}\{1+\vartheta_{k}(t)\}. \quad (2.24)$$

Analogous results for the derivative are expressed by

$$\mathrm{d}\{t^{\frac{1}{4}(m-2)}U_j(t)\}/\mathrm{d}t = \tfrac{1}{2}m\mathrm{i}^{-k-1}\mathrm{e}^{-j\pi\mathrm{i}/m}\lambda_{j,k}\, t^{\frac{1}{2}(m-2)} \exp\{(t\,\mathrm{e}^{-2k\pi\mathrm{i}/m})^{\frac{1}{2}m}\}\{1+\vartheta^{1}_{k\pm 1}(t)\}$$
$$\mp \tfrac{1}{2}m\mathrm{i}^{-k}\mathrm{e}^{-j\pi\mathrm{i}/m}\lambda_{j,k\pm 1}\, t^{\frac{1}{2}(m-2)} \exp\{-(t\,\mathrm{e}^{-2k\pi\mathrm{i}/m})^{\frac{1}{2}m}\}\{1+\vartheta^{1}_{k}(t)\}, \quad (2.25)$$

the upper signs being taken when $t$ lies in the left part of $\boldsymbol{S}_k$ (or the right part of $\boldsymbol{S}_{k+1}$), and the lower signs being taken when $t$ lies in the right part of $\boldsymbol{S}_k$ (or the left part of $\boldsymbol{S}_{k-1}$).

### 2.5. *Auxiliary functions*

Throughout this subsection we assume that the conditions (2.19) and (2.20) are fulfilled, ensuring that $\lambda_{j,k} \neq 0$.

From (2.12), (2.23) and (2.24), we see that as $t \to \infty$ the solution $U_j(t)$ is recessive in $\boldsymbol{S}_j$ and dominant in $\boldsymbol{S}_k$; by symmetry $U_k(t)$ is dominant in $\boldsymbol{S}_j$ and recessive in $\boldsymbol{S}_k$. In consequence, $U_j(t)$ and $U_k(t)$ comprise a numerically satisfactory pair of solutions in the closed region $\boldsymbol{S}_j \cup \boldsymbol{S}_k$, except possibly in the neighbourhood of $t = 0$. For our purposes $U_j(t)$ and $U_k(t)$ comprise an appropriate solution basis in $\boldsymbol{S}_j \cup \boldsymbol{S}_k$, and in order to majorize these solutions in a satisfactory manner, we follow the procedure of § 8.3 of Olver (1974, ch. 11), and introduce auxiliary weight, modulus and phase functions, as follows.

We define a function $e(t)$ by the formula

$$e(t) = |\exp\{(-)^j t^{\frac{1}{2}m}\}| \quad (t \in \boldsymbol{S}_j), \quad (2.26)$$

for every integer $j$, where the branch of $t^{\frac{1}{2}m}$ is $|t|^{\frac{1}{2}m} \exp(\frac{1}{2}m\mathrm{i}\,\mathrm{ph}\, t)$. On the boundaries of $\boldsymbol{S}_j$ we have $e(t) = 1$, hence $e(t)$ is continuous everywhere. Furthermore, $e(t) \geqslant 1$. As weight function in $\boldsymbol{S}_j \cup \boldsymbol{S}_k$ we shall adopt $E_{j,k}(t)$, defined by

$$E_{j,k}(t) = 1/e(t) \quad (t \in \boldsymbol{S}_j); \quad E_{j,k}(t) = e(t) \quad (t \in \boldsymbol{S}_k). \quad (2.27)$$

Clearly $E_{j,k}$ is continuous, and $E_{j,k}(0) = 1$. Also, with $E^{-1}_{j,k}(t)$ denoting $1/E_{j,k}(t)$, we have

$$E_{k,j}(t) = E^{-1}_{j,k}(t) \quad (t \in \boldsymbol{S}_j \cup \boldsymbol{S}_k). \quad (2.28)$$

Modulus and phase functions are defined in $\boldsymbol{S}_j \cup \boldsymbol{S}_k$ by the equations

$$|U_j(t)| = E_{j,k}(t)\, M_{j,k}(t) \cos\theta_{j,k}(t), \quad |U_k(t)| = E^{-1}_{j,k}(t)\, M_{j,k}(t) \sin\theta_{j,k}(t), \quad (2.29)$$
$$|U'_j(t)| = E_{j,k}(t)\, N_{j,k}(t) \cos\omega_{j,k}(t), \quad |U'_k(t)| = E^{-1}_{j,k}(t)\, N_{j,k}(t) \sin\omega_{j,k}(t). \quad (2.30)$$

In consequence

$$M_{j,k}(t) = \{E_{j,k}^{-2}(t)\,|U_j(t)|^2 + E_{j,k}^2(t)\,|U_k(t)|^2\}^{\frac{1}{2}},$$

$$N_{j,k}(t) = \{E_{j,k}^{-2}(t)\,|U_j'(t)|^2 + E_{j,k}^2(t)\,|U_k'(t)|^2\}^{\frac{1}{2}},$$

$$\theta_{j,k}(t) = \arctan\{E_{j,k}^2(t)\,|U_k(t)/U_j(t)|\},$$

$$\omega_{j,k}(t) = \arctan\{E_{j,k}^2(t)\,|U_k'(t)/U_j'(t)|\}.$$

Each of the functions $M_{j,k}(t)$, $\theta_{j,k}(t)$, $N_{j,k}(t)$ and $\omega_{j,k}(t)$ is continuous in $\boldsymbol{S}_j \cup \boldsymbol{S}_k$, except that $N_{j,k}(t)$ has an infinity at $t = 0$ when $0 < m \leqslant 1$. Symmetry properties follow from (2.28), and are given by

$$M_{j,k}(t) = M_{k,j}(t), \quad \theta_{j,k}(t) + \theta_{k,j}(t) = \tfrac{1}{2}\pi, \tag{2.31}$$

$$N_{j,k}(t) = N_{k,j}(t), \quad \omega_{j,k}(t) + \omega_{k,j}(t) = \tfrac{1}{2}\pi. \tag{2.32}$$

At the origin

$$M_{j,k}(0) = 2^{\frac{1}{2}}|U(0)|, \quad \theta_{j,k}(0) = \omega_{j,k}(0) = \tfrac{1}{4}\pi, \tag{2.33}$$

$$N_{j,k}(0) = 2^{\frac{1}{2}}|U'(0)| \quad (m > 1); \quad N_{j,k}(0) = \infty \quad (0 < m \leqslant 1). \tag{2.34}$$

The asymptotic behaviour of the modulus functions for large $|t|$ may be deduced from (2.12), (2.13), (2.23), (2.24) and (2.25). In internal parts of $\boldsymbol{S}_j$ and $\boldsymbol{S}_k$ we find that

$$M_{j,k}(t) \sim (1+\lambda_{j,k}^2)^{\frac{1}{2}}|t|^{\frac{1}{4}(2-m)}, \tag{2.35}$$

$$N_{j,k}(t) \sim \tfrac{1}{2}m(1+\lambda_{j,k}^2)^{\frac{1}{2}}|t|^{\frac{1}{4}(m-2)}. \tag{2.36}$$

And in these circumstances the phase functons $\theta_{j,k}(t)$ and $\omega_{j,k}(t)$ tend to constant values, both values being $\arctan|\lambda_{j,k}|$ in internal parts of $\boldsymbol{S}_j$ and $\operatorname{arccot}|\lambda_{j,k}|$ in internal parts of $\boldsymbol{S}_k$.

In the full domain $\boldsymbol{S}_j \cup \boldsymbol{S}_k$ the modulus and phase functions fluctuate as $t \to \infty$. In the case of $M_{j,k}(t)$ we shall employ the following uniform bound, again derived from (2.12), (2.23) and (2.24):

$$M_{j,k}(t) \leqslant C_{j,k}\,|t|^{\frac{1}{4}(2-m)}\{1+\mathcal{O}(t^{\frac{1}{2}m})\} \quad (t \in \boldsymbol{S}_j \cup \boldsymbol{S}_k), \tag{2.37}$$

where

$$C_{j,k} = \max[\{1+(|\lambda_{j,k}|+|\lambda_{j,k+1}|)^2\}^{\frac{1}{2}}, \quad \{1+(|\lambda_{j,k}|+|\lambda_{j,k-1}|)^2\}^{\frac{1}{2}}]. \tag{2.38}$$

No such simple bound is available for $N_{j,k}(t)$, and for this reason we shall sometimes find it convenient to work in terms of another pair of modulus and phase functions, defined by

$$|\mathrm{d}\{t^{\frac{1}{4}(m-2)}U_j(t)\}/\mathrm{d}t| = |t|^{\frac{1}{4}(m-2)}E_{j,k}(t)\,\hat{N}_{j,k}(t)\cos\hat{\omega}_{j,k}(t), \tag{2.39}$$

$$|\mathrm{d}\{t^{\frac{1}{4}(m-2)}U_k(t)\}/\mathrm{d}t| = |t|^{\frac{1}{4}(m-2)}E_{j,k}^{-1}(t)\,\hat{N}_{j,k}(t)\sin\hat{\omega}_{j,k}(t). \tag{2.40}$$

Some of the corresponding properties of $\hat{N}_{j,k}(t)$ and $\hat{\omega}_{j,k}(t)$ are given by

$$\left.\begin{aligned} \hat{N}_{j,k}(t) &= |t|^{\frac{1}{4}(2-m)}[E_{j,k}^{-2}(t)\,|\mathrm{d}\{t^{\frac{1}{4}(m-2)}U_j(t)\}/\mathrm{d}t|^2 + E_{j,k}^2(t)\,|\mathrm{d}\{t^{\frac{1}{4}(m-2)}U_k(t)\}/\mathrm{d}t|^2]^{\frac{1}{2}},\\ \hat{\omega}_{j,k}(t) &= \arctan[E_{j,k}^2(t)\,|\mathrm{d}\{t^{\frac{1}{4}(m-2)}U_k(t)\}/\mathrm{d}t|/|\mathrm{d}\{t^{\frac{1}{4}(m-2)}U_j(t)\}/\mathrm{d}t|],\\ \hat{N}_{j,k}(t) &= \hat{N}_{k,j}(t), \quad \hat{\omega}_{j,k}(t)+\hat{\omega}_{k,j}(t) = \tfrac{1}{2}\pi, \quad \hat{\omega}_{j,k}(0) = \tfrac{1}{4}\pi, \end{aligned}\right\} \tag{2.41}$$

and

$$\hat{N}_{j,k}(t) \sim \tfrac{1}{2}m(1+\lambda_{j,k}^2)^{\frac{1}{2}}|t|^{\frac{1}{4}(m-2)} \tag{2.42}$$

as $t \to \infty$ in internal parts of $\boldsymbol{S}_j$ and $\boldsymbol{S}_k$. Lastly,

$$\hat{N}_{j,k}(t) \leqslant \tfrac{1}{2}mC_{j,k}|t|^{\frac{1}{4}(m-2)}\{1+\mathcal{O}(t^{\frac{1}{2}m})\} \quad (t \in \boldsymbol{S}_j \cup \boldsymbol{S}_k). \tag{2.43}$$

## 3. Main approximation theorem

### 3.1. *Primary assumptions*

In this section we seek approximate solutions of the differential equation

$$\mathrm{d}^2w/\mathrm{d}z^2 = \{u^2f(u,z)+g(u,z)\}\,w, \tag{3.1}$$

in which $u$ is a positive parameter and $z$ ranges over a bounded or unbounded open complex domain $\boldsymbol{D}$, which may depend on $u$. We suppose $c$ to be a given interior point of $\boldsymbol{D}$, which may depend on $u$, and $m$ to be a positive constant. Then $(z-c)^{2-m}f(u,z)$ is assumed to be holomorphic and non-vanishing throughout $\boldsymbol{D}$, including $c$. The function $g(u,z)$, also, is assumed to be holomorphic in $\boldsymbol{D}$ except possibly at $c$; at this point we require

$$g(u,z) = O\{(z-c)^{\gamma-1}\} \quad (z\to c), \tag{3.2}$$

where $\gamma$ is a positive constant. (Condition (3.2) suffices for our present purposes, but it could be eased somewhat without affecting the final conclusions.)

Because $f(u,z)$ and $g(u,z)$ may have branch-points at $c$, we do not confine $\boldsymbol{D}$ to a single Riemann sheet. Any number of sheets may be used, provided that $\boldsymbol{D}$ is simply connected on the aggregate Riemann surface.

### 3.2. *Preliminary definitions and transformations*

Following the treatment of the real-variable case given in Olver (1977*a*, §3.1), we introduce a new variable $\zeta=\zeta(u,z)$, defined by

$$\zeta = \left\{\int_c^z f^{\frac{1}{2}}(u,t)\,\mathrm{d}t\right\}^{2/m}. \tag{3.3}$$

The branches of the fractional powers in this relation are determined as follows. Let the Taylor-series expansion of $(z-c)^{2-m}f(u,z)$ in the neighbourhood of $c$ be denoted by

$$(z-c)^{2-m}f(u,z) = f_0+f_1(z-c)+f_2(z-c)^2+\dots, \tag{3.4}$$

where $f_0\neq 0$. Substituting for $f(u,z)$ in (3.3) and integrating term by term, we find that $\zeta$ has a Taylor-series expansion that begins

$$\zeta = \left(\frac{2}{m}\right)^{2/m} f_0^{1/m}(z-c)\left\{1+\frac{1}{m+2}\frac{f_1}{f_0}(z-c)+\dots\right\}. \tag{3.5}$$

We select any branch of the coefficient $f_0^{1/m}$ that is convenient. This fixes the relation between $\zeta$ and $z$ in the neighbourhood of $c$; elsewhere $\zeta$ is determined by continuity.

Having prescribed $\zeta(u,z)$ in a unique manner we define $\boldsymbol{\Delta}$ to be the map of $\boldsymbol{D}$ on the $\zeta$-plane. Like $\boldsymbol{D}$, $\boldsymbol{\Delta}$ is a domain that may lie on an enumerable set of Riemann sheets. We suppose, however, that $\boldsymbol{D}$ *is restricted in such a way that the mapping from* $\boldsymbol{D}$ *on to* $\boldsymbol{\Delta}$ *is one-to-one*. In particular, this means that the only point of $\boldsymbol{D}$ at which the right hand side of (3.3) may vanish is $c$. And since the only possible zero or singularity of $f(u,z)$ is located at $c$, it follows that $\zeta(u,z)$ is holomorphic in $\boldsymbol{D}$, and therefore that the mapping from $\boldsymbol{D}$ on to $\boldsymbol{\Delta}$ is conformal.

Next, for every integer $j$ we define $\boldsymbol{D}_j$ to be the map on the $z$-plane of $\boldsymbol{\Delta}\cap\boldsymbol{S}_j$, where $\boldsymbol{S}_j$ is the sector defined in §2.2. We call $\boldsymbol{D}_j$ a *principal region associated with the transition point c*. (Although $\boldsymbol{D}_j$ is simply connected, it is not necessarily connected, and therefore may not be a domain in the ordinary sense.)


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A simple illustration is provided by taking

$$f(u,z) = z^2/(1-z)^6, \quad g(u,z) = 0,$$

with $\boldsymbol{D}$ comprising the $z$-plane after removal of the interval $[1,\infty)$. In this case equation (3.1) has a double turning point at the origin and an irregular singularity at $z = 1$. From (3.3) with $c = 0$ and $m = 4$, we find that $\zeta$ is given in terms of $z$ by the fractional linear transformation

$$\zeta = z/\{2^{\frac{1}{2}}(1-z)\}.$$

Corresponding regions of the mapping are indicated in figures 2 and 3. The turning point remains at the origin, and the points $z = 1$ and $\infty$ are projected to $\zeta = \infty$ and $-2^{-\frac{1}{2}}$, respectively. Thus $\varDelta$ comprises the whole $\zeta$-plane with the interval $(-\infty, -2^{-\frac{1}{2}}]$ deleted. The mapping is one-to-one, as required, and the boundaries of the principal regions are circular arcs and also, in the case of $\boldsymbol{D}_2$, the interval $[1,\infty)$.

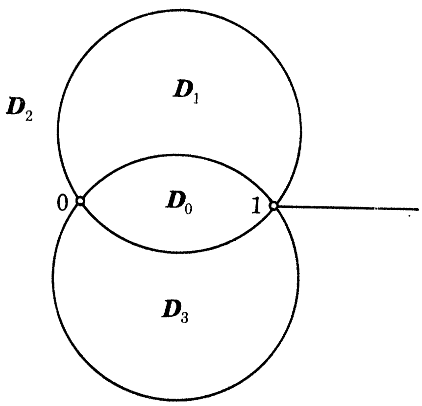


FIGURE 2. $z$-plane.

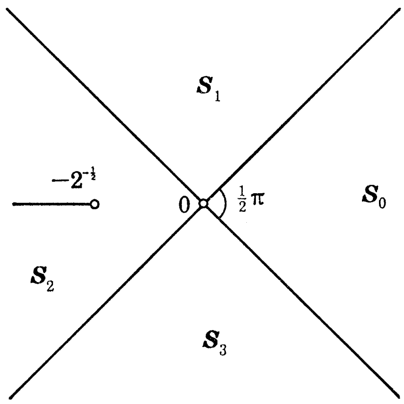


FIGURE 3. $\zeta$-plane.

Returning to the general problem, we continue to follow Olver (1977*a*, §3.1), and define

$$\hat{f}(u,z) = \frac{4f(u,z)}{m^2\zeta^{m-2}} = \left(\frac{\mathrm{d}\zeta}{\mathrm{d}z}\right)^2; \tag{3.6}$$

compare (3.3). Evidently $\hat{f}(u,z)$ is holomorphic and non-vanishing in $\boldsymbol{D}$.

Next, we introduce a balancing function $\Omega(t)$. This is any convenient positive real function of the complex variable $t$ that is continuous and satisfies the condition

$$\Omega(t) = O(t^{\frac{1}{2}(m-2)}), \tag{3.7}$$

uniformly in the neighbourhood of $t = \infty$. An admissible choice is given by†

$$\Omega(t) = (1+|t|^{\frac{1}{2}m})/(1+|t|). \tag{3.8}$$

Associated with $\Omega(t)$ we define $\rho_{j,k}$ to be the constant

$$\rho_{j,k} = \sup_{t\in \boldsymbol{S}_j\cup \boldsymbol{S}_k} \{\Omega(t)\, M^2_{j,k}(t)\}. \tag{3.9}$$

In consequence of (2.37), (3.7) and the fact that $M_{j,k}(t)$ is continuous in $\boldsymbol{S}_j \cup \boldsymbol{S}_k$, the value of $\rho_{j,k}$ is always finite.

† The choice $1+|t|^{\frac{1}{2}(m-2)}$ that was used in the real-variable case is also suitable here when $m \geqslant 2$, but not when $0 < m < 2$ because of discontinuous behaviour at the origin.

Lastly, we introduce the *error-control function*

$$H(u,z) \equiv \int\left\{\frac{1}{f^{\frac{1}{4}}(u,z)}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\left(\frac{1}{f^{\frac{1}{4}}(u,z)}\right)-\frac{g(u,z)}{f^{\frac{1}{2}}(u,z)}\right\}\frac{\mathrm{d}z}{\Omega(u^{2/m}\zeta)}. \tag{3.10}$$

Any branches of $f^{\frac{1}{4}}(u,z)$ and $f^{\frac{1}{2}}(u,z)$ may be used, provided they are continuous in $\boldsymbol{D}$ and the latter is the square of the former. The choice of integration constant is immaterial.

### 3.3. *Statement of theorem*

THEOREM 3.1. *With the assumptions of* § 3.1 *and the definitions of* §§ 2 *and* 3.2, *let $A$ and $B$ be arbitrary real or complex constants, and $a_j$ be an arbitrarily chosen reference point in the closure of $\boldsymbol{D}_j$, including the point at infinity. Let $\boldsymbol{H}_k(a_j)$ comprise the set of points $z$ in $\boldsymbol{D}_j \cup \boldsymbol{D}_k$ that can be joined to $a_j$ by a path*† $\mathscr{P}$ *that lies in $\boldsymbol{D}_j \cup \boldsymbol{D}_k$ and satisfies the following monotonicity condition: as $t$ travels along $\mathscr{P}$ from $a_j$ to $z$ the real part of*

$$\xi(u,z) \equiv \int_c^z f^{\frac{1}{2}}(u,t)\,\mathrm{d}t \tag{3.11}$$

*is non-decreasing, where the branch of this integral is continuous and has non-positive real part in $\boldsymbol{D}_j$ and non-negative real part in $\boldsymbol{D}_k$. Then with conditions* (i), (ii) *and* (iii) *given below, equation* (3.1) *has a solution $w(u,z)$ that depends on $A$, $B$ and $a_j$, is holomorphic in $\boldsymbol{D}$ punctured at $c$, and is continuous at $c$. Furthermore, when $z \in \boldsymbol{H}_k(a_j)$*

$$w(u,z) = f^{-\frac{1}{4}}(u,z)\,\{AU_j(u^{2/m}\zeta)+BU_k(u^{2/m}\zeta)+\epsilon(u,\zeta)\}, \tag{3.12}$$

*where*‡

$$\frac{|\epsilon(u,\zeta)|}{M_{j,k}(u^{2/m}\zeta)},\quad \frac{|\,\partial\epsilon(u,\zeta)/\partial\zeta|}{u^{2/m}N_{j,k}(u^{2/m}\zeta)},\quad \frac{|\partial\{\zeta^{\frac{1}{4}(m-2)}\epsilon(u,\zeta)\}/\partial\zeta|}{u^{2/m}|\zeta|^{\frac{1}{4}(m-2)}\hat{N}_{j,k}(u^{2/m}\zeta)}$$
$$\leqslant \frac{\sigma_{j,k}}{\rho_{j,k}}E_{j,k}(u^{2/m}\zeta)\left[\exp\left\{\frac{\rho_{j,k}}{m|\lambda_{j,k}|u^{2/m}}\mathscr{V}_{a_j,z}(H)\right\}-1\right], \tag{3.13}$$

*and*
$$\sigma_{j,k} = \sup_{v\in\mathscr{Q}}\{\Omega(u^{2/m}v)\,E_{j,k}^{-1}(u^{2/m}v)\,M_{j,k}(u^{2/m}v)\,|AU_j(u^{2/m}v)+BU_k(u^{2/m}v)|\}, \tag{3.14}$$

*$\lambda_{j,k}$ being defined by* (2.17) *and* (2.18), *$\mathscr{Q}$ denoting the $\zeta$-map of $\mathscr{P}$, and the variation of $H$ in* (3.13) *being evaluated along $\mathscr{P}$.*

The supplementary conditions are as follows:

(i) $k$ is an integer other than $j$. If $m$ is a rational number of the form $m_1/m_2$ where $m_1 \neq 1$ and $m_1$ and $m_2$ are mutually prime positive integers, then we require $|k-j| < m_1$. For all other values of $m$, $k$ is unrestricted.

(ii) If the $\zeta$-map of $a_j$ is at infinity, then we suppose that it is the point at infinity on a path $\mathscr{L}$ in $\boldsymbol{S}_j$ and require $\mathscr{P}$ to coincide with $\mathscr{L}$ in the neighbourhood of $a_j$.

(iii) When $z \in \boldsymbol{D}_k$ the bounds (3.13) apply only to the branch of $w(u,z)$ obtained by analytic continuation from $\boldsymbol{D}_j$ in the neighbourhood of $c$ by rotation through an angle $2(k-j)\,\pi/m$, the sense of this angle being the same as the sign of $k-j$.

*Remarks.* (*a*) Any path $\mathscr{P}$ that satisfies the monotonicity condition of the theorem will be called a *progressive path*. It should be observed that when $z \in \boldsymbol{D}_k$ and $k-j = \pm 1$ it is possible to satisfy the monotonicity condition without the need for $\mathscr{P}$ to pass through $c$.

† Strictly, by 'path' we mean a Jordan arc composed of a finite chain of $R_2$ arcs in the sense defined in § 3.4 of Olver (1974, ch. 5).

‡ The symbol $\mathscr{V}$ denotes the variational operator as defined in § 11.5 of Olver (1974, ch. 1).

(*b*) For the theorem to be meaningful it is necessary that the right hand side of (3.13) be finite. In the first place this condition requires $\mathscr{V}_{\mathscr{P}}(H)$ to converge. Next, if $k$ and $j$ satisfy condition (i), then it is easily verified that (2.19) and (2.20) are satisfied, ensuring $1/|\lambda_{j,k}|$ is finite. It is also essential that $\sigma_{j,k}$ be finite. From the properties of the auxiliary functions given in §2.5 and the assumed property (3.7) of the balancing function, it is seen that the content of the braces in (3.14) is finite at all finite points $v$ in $\boldsymbol{S}_j \cup \boldsymbol{S}_k$, and bounded as $v \to \infty$ in $\boldsymbol{S}_k$. As $v \to \infty$ in an internal part of $\boldsymbol{S}_j$, however, the content of the braces in (3.14) is unbounded, unless $B = 0$. Accordingly, *if the $\zeta$-map of $a_j$ lies at infinity in an internal part of $\boldsymbol{S}_j$ then the theorem may be applied only in the case* $B = 0$.

(*c*) The asymptotic nature of the error bounds (3.13) for large values of $u$ is investigated in section 4.

### 3.4. *Proof when c is a multiple turning point*

Throughout this subsection we suppose that $m$ is a positive integer other than 1, and $g(u, z)$ is analytic at $c$. Thus the differential equation (3.1) has a turning point of multiplicity $m-2$ at $c$, and all solutions are analytic at $c$ (as well as at all other points of $\boldsymbol{D}$).

On introducing a new dependent variable $W$ given by

$$w = \dot{z}^{\frac{1}{2}} W, \tag{3.15}$$

where the dot signifies differentiation with respect to $\zeta$, we have a Liouville transformation of (3.1) from the variables $w$ and $z$ to the variables $W$ and $\zeta$. The transformed equation is given by

$$\mathrm{d}^2W/\mathrm{d}\zeta^2 = \{\tfrac{1}{4}m^2u^2\zeta^{m-2} + \phi(u,\zeta)\}\, W, \tag{3.16}$$

where

$$\phi(u,\zeta) = \dot{z}^{\frac{1}{2}}\{\mathrm{d}^2(\dot{z}^{-\frac{1}{2}})/\mathrm{d}\zeta^2\} + \dot{z}^2 g(u,z). \tag{3.17}$$

In the present circumstances $g(u, z)$ is holomorphic in $\boldsymbol{D}$. We also know that $z \equiv z(u, \zeta)$ is a holomorphic function of $\zeta$ in $\boldsymbol{\Delta}$ and $\dot{z}$ is non-vanishing. Hence $\phi(u, \zeta)$ is holomorphic in $\boldsymbol{\Delta}$.

From (3.6), (3.12) and (3.15) we see that

$$W = AU_j(u^{2/m}\zeta) + BU_k(u^{2/m}\zeta) + \epsilon(u,\zeta). \tag{3.18}$$

Next, from (2.1) with $t = u^{2/m}\zeta$ we derive

$$\mathrm{d}^2\{AU_j(u^{2/m}\zeta) + BU_k(u^{2/m}\zeta)\}/\mathrm{d}\zeta^2 = \tfrac{1}{4}m^2u^2\zeta^{m-2}\{AU_j(u^{2/m}\zeta) + BU_k(u^{2/m}\zeta)\}. \tag{3.19}$$

On substituting in (3.16) by means of (3.18) and then subtracting (3.19), we arrive at the following inhomogeneous differential equation for $\epsilon(u, \zeta)$:

$$\mathrm{d}^2\epsilon/\mathrm{d}\zeta^2 - \tfrac{1}{4}m^2u^2\zeta^{m-2}\epsilon = \phi(u,\zeta)\,\{AU_j(u^{2/m}\zeta) + BU_k(u^{2/m}\zeta) + \epsilon\}. \tag{3.20}$$

An equivalent integral equation is obtained by applying the method of variation of parameters and using the Wronskian relation (2.16); thus

$$\epsilon(u,\zeta) = \frac{\mathrm{i}}{mu^{2/m}}\int_{\alpha_j}^{\zeta} K(\zeta,v)\,\phi(u,v)\,\{AU_j(u^{2/m}v) + BU_k(u^{2/m}v) + \epsilon(u,v)\}\,\mathrm{d}v, \tag{3.21}$$

where $\zeta = \alpha_j$ corresponds to the reference point $a_j$, and

$$K(\zeta,v) = \mathrm{e}^{(j+k)\pi\mathrm{i}/m}\lambda_{j,k}^{-1}\{U_j(u^{2/m}\zeta)\,U_k(u^{2/m}v) - U_k(u^{2/m}\zeta)\,U_j(u^{2/m}v)\}. \tag{3.22}$$

The integration path in (3.21) is taken to be the $\zeta$-map $\mathscr{Q}$ of $\mathscr{P}$. It therefore lies in $\boldsymbol{S}_j \cup \boldsymbol{S}_k$.


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From (2.26), (2.27) and the definition of $\xi(u,z)$ it is seen that $E_{j,k}(u^{2/m}\zeta) = |e^{u\xi}|$. Hence in consequence of the monotonicity condition we have

$$E_{j,k}(u^{2/m}v) \leqslant E_{j,k}(u^{2/m}\zeta) \quad (v \in \mathscr{Q}). \tag{3.23}$$

Substituting in (3.22) by means of (2.29) and using (3.23), we find that

$$|K(\zeta,v)| \leqslant |\lambda_{j,k}^{-1}| E_{j,k}(u^{2/m}\zeta)\, E_{j,k}^{-1}(u^{2/m}v)\, M_{j,k}(u^{2/m}\zeta)\, M_{j,k}(u^{2/m}v). \tag{3.24}$$

Similarly,

$$|\partial K(\zeta,v)/\partial\zeta| \leqslant |\lambda_{j,k}^{-1}|\, u^{2/m} E_{j,k}(u^{2/m}\zeta)\, E_{j,k}^{-1}(u^{2/m}v)\, N_{j,k}(u^{2/m}\zeta)\, M_{j,k}(u^{2/m}v). \tag{3.25}$$

The integral equation (3.21) may now be solved by successive approximation, for example, by application of theorem 10.2 of Olver (1974, ch. 6). Provided that $\int\{|\phi(u,v)|/\Omega(u^{2/m}v)\}|dv|$ converges as $v \to \alpha_j$ along $\mathscr{Q}$ – and this will be justified shortly – we conclude that there is a unique solution $\epsilon(u,v)$ that is twice continuously differentiable along $\mathscr{Q}$ and enjoys the properties

$$\frac{\epsilon(u,v)}{E_{j,k}(u^{2/m}v)\, M_{j,k}(u^{2/m}v)}, \quad \frac{\partial\epsilon(u,v)/\partial v}{E_{j,k}(u^{2/m}v)\, N_{j,k}(u^{2/m}v)} \to 0$$

as $v \to \alpha_j$ along $\mathscr{Q}$. Furthermore, when $v \in \mathscr{Q}$

$$\frac{|\epsilon(u,v)|}{E_{j,k}(u^{2/m}v)\, M_{j,k}(u^{2/m}v)}, \quad \frac{|\partial\epsilon(u,v)/\partial v|}{u^{2/m}E_{j,k}(u^{2/m}v)\, N_{j,k}(u^{2/m}v)} \leqslant \frac{\sigma_{j,k}}{\rho_{j,k}}\left[\exp\left\{\frac{\rho_{j,k}}{m|\lambda_{j,k}|u^{2/m}}\int_{\alpha_j}^{v}\frac{|\phi(u,\tau)|}{\Omega(u^{2/m}\tau)}|d\tau|\right\} - 1\right], \tag{3.26}$$

where $\rho_{j,k}$ and $\sigma_{j,k}$ are defined by (3.9) and (3.14), respectively. In particular, these inequalities hold when $v = \zeta$. We now transform back to the $z$-plane in the manner of § 11.2 of Olver (1974, ch. 6) using the identity

$$H(u,z) = -\int \frac{\phi(u,\zeta)}{\Omega(u^{2/m}\zeta)}\, d\zeta, \tag{3.27}$$

obtained from (3.6), (3.10) and (3.17). This yields the first and second of the inequalities (3.13); it also establishes the convergence of the integral within the braces in (3.26) since $\mathscr{V}_{\mathscr{P}}(H)$ converges, by hypothesis.

To establish the remaining inequality in (3.13), we use the bound

$$|\partial\{\zeta^{\frac{1}{4}(m-2)}K(\zeta,v)\}/\partial\zeta| \leqslant |\lambda_{j,k}^{-1}|u^{2/m}|\zeta|^{\frac{1}{4}(m-2)}E_{j,k}(u^{2/m}\zeta)\, E_{j,k}^{-1}(u^{2/m}v)\, \hat{N}_{j,k}(u^{2/m}\zeta)\, M_{j,k}(u^{2/m}v), \tag{3.28}$$

obtained from (3.22) with the aid of (2.39) and (2.40).

*Remark.* With the conditions adopted in the opening paragraph of this subsection passage around any closed circuit in $\boldsymbol{D}$ leaves the solution $w(u,z)$ unchanged. Also, $M_{j,k+m}(u^{2/m}\zeta)$ is the same as $M_{j,k}(u^{2/m}\zeta)$, and so on. Hence in the present circumstances condition (i) of § 3.3 may be eased to $k \neq j \pmod m$.

### 3.5. *Proof in the general case: preliminaries*

When $m$ is permitted to assume positive values other than 2, 3, 4, ..., and $g(u,z)$ may be singular at $c$, the analysis of § 3.4 can be retraced until we reach the integral equation (3.21), with the modification that $\phi(u,\zeta)$ may be singular at $\zeta = 0$. As we saw in §§ 2.1 and 2.2, $U_j(u^{2/m}\zeta)$ is continuous at $\zeta = 0$, but its derivative $U_j'(u^{2/m}\zeta)$ is infinite at this point for certain values of $m$. Similarly for $U_k(u^{2/m}\zeta)$. Hence although $\epsilon(u,v)$ represents a continuously differentiable solution of (3.20) on $\mathscr{Q} \cap \boldsymbol{S}_j$ and another continuously differentiable solution on $\mathscr{Q} \cap \boldsymbol{S}_k$, because of the discontinuity in $\partial\epsilon(u,v)/\partial v$ at $v = 0$ it is no longer obvious how the two solutions are related.

In order to resolve this difficulty we now suppose that $\mathscr{Q}$ denotes an arbitrary path in $\Delta \cap (\boldsymbol{S}_j \cup \boldsymbol{S}_k)$ that begins at $\alpha_j$ and terminates at an arbitrary point $\beta_k$ in $\boldsymbol{S}_k$. We shall suppose that $E_{j,k}(u^{2/m}v)$ is non-decreasing as $v$ travels along $\mathscr{Q}$ from $\alpha_j$ to $\beta_k$; in other words, $\mathscr{Q}$ has to be the map of a progressive path in the sense defined in remark ($a$) of §3.3. Next, we denote by $\boldsymbol{S}_{j,k}$ the union of $\boldsymbol{S}_j$, $\boldsymbol{S}_k$ and all intervening sectors, that is, $\boldsymbol{S}_{j+1}, \boldsymbol{S}_{j+2}, \ldots, \boldsymbol{S}_{k-1}$ if $k > j$, or $\boldsymbol{S}_{j-1}, \boldsymbol{S}_{j-2}, \ldots, \boldsymbol{S}_{k+1}$ if $k < j$. Instead of solving (3.21) directly on $\mathscr{Q}$, we first indent $\mathscr{Q}$ in the neighbourhood of the origin by a circular arc of radius $\varpi^{2/m}$, where $\varpi$ is a positive constant that is sufficiently small to ensure that this indentation is contained in $\Delta \cap \boldsymbol{S}_{j,k}$. The indented path will be denoted by $\mathscr{Q}_\varpi$. As we travel along $\mathscr{Q}_\varpi$ from $\alpha_j$ to $\beta_k$, we suppose that the origin is passed in the positive or negative rotational sense according as $k > j$ or $k < j$; compare figure 4.

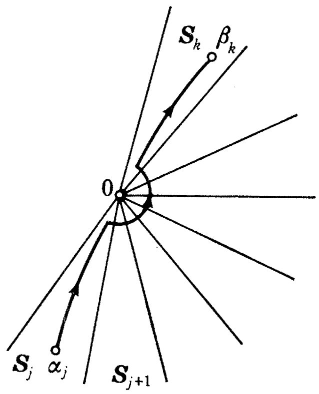


FIGURE 4. $\zeta$-plane: path $\mathscr{Q}_\varpi$ when $k > j$.

We shall need another expression for the kernel of the integral equation (3.21), given by

$$K(\zeta, v) = e^{(p+q)\pi i/m}\lambda_{p,q}^{-1}\{U_p(u^{2/m}\zeta)\,U_q(u^{2/m}v) - U_q(u^{2/m}\zeta)\,U_p(u^{2/m}v)\}. \tag{3.29}$$

Here $p$ and $q$ are any two integers such that $\lambda_{p,q} \neq 0$. This formula is obtainable from (2.21) and (3.22) by straightforward substitutions.

We shall also need to associate certain auxiliary functions with $\boldsymbol{S}_{j,k}$ in addition to those of §2.5. First, we define a weight function $\bar{E}_{j,k}(t)$ by the formulae

$$\bar{E}_{j,k}(t) = 1/e(t) \quad (t \in \boldsymbol{S}_j); \quad \bar{E}_{j,k}(t) = e(t) \quad (t \notin \boldsymbol{S}_j).$$

Clearly $\bar{E}_{j,k}(t)$ is continuous in $\boldsymbol{S}_{j,k}$. Secondly, we denote by $\{j,k\}$ the set of all integers from $j$ to $k$, inclusive. Then for every $p \in \{j,k\}$ and $t \in \boldsymbol{S}_p$ we define

$$\bar{M}_{j,k}(t) = \max_{q \neq p}\{M_{p,q}(t)\} \quad (q \in \{j,k\}).$$

Since each modulus function $M_{p,q}(t)$ is continuous in $\boldsymbol{S}_p$ it is easily seen that $\bar{M}_{j,k}(t)$ is continuous in $\boldsymbol{S}_{j,k}$, except possibly on the boundaries of the sectors that intervene between $\boldsymbol{S}_j$ and $\boldsymbol{S}_k$. In order to prove that $\bar{M}_{j,k}(t)$ is continuous on these boundaries, we assume temporarily that $k > j$, and let $t$ be any point on the left boundary of $\boldsymbol{S}_p$, where $j \leqslant p \leqslant k-1$. On referring to the definition (2.10) and rotating the $t$-plane, we see that $U_p(t) = U_{p+1}(t')$, where $t'$ is the point on the left boundary of $\boldsymbol{S}_{p+1}$ such that $|t'| = |t|$. Again, from (2.10) and the fact that $U(t)$ is real on the positive real $t$-axis it follows that $U_{p+1}(t)$ and $U_{p+1}(t')$ are complex conjugates. Hence $|U_p(t)| = |U_{p+1}(t)|$. This implies that $\bar{M}_{j,k}(t)$ has the same value whether $t$ be counted as a member of

$\boldsymbol{S}_p$ or $\boldsymbol{S}_{p+1}$. Therefore $\overline{M}_{j,k}(t)$ is continuous on the boundary $\boldsymbol{S}_p \cap \boldsymbol{S}_{p+1}$. Similar analysis may also be used when $k < j$, and we therefore conclude that $\overline{M}_{j,k}(t)$ *is continuous throughout* $\boldsymbol{S}_{j,k}$.

In an analogous manner, for every $p \in \{j, k\}$ and $t \in \boldsymbol{S}_p$ we define

$$\overline{N}_{j,k}(t) = \max_{q \neq p} \{N_{p,q}(t)\} \quad (q \in \{j, k\}).$$

By similar analysis we see that $\overline{N}_{j,k}(t)$ is continuous in $\boldsymbol{S}_{j,k}$, except at $t = 0$ when $0 < m \leqslant 1$.

From the definitions of $\overline{M}_{j,k}(t)$ and $\overline{N}_{j,k}(t)$ it is immediately clear that

$$\overline{M}_{j,k}(t) = \overline{M}_{k,j}(t), \quad \overline{N}_{j,k}(t) = \overline{N}_{k,j}(t). \tag{3.30}$$

Also, from the asymptotic properties of $M_{j,k}(t)$ and $N_{j,k}(t)$ given in §2.5, we derive

$$\overline{M}_{j,k}(t) = O(t^{\frac{1}{4}(2-m)}), \quad \overline{N}_{j,k}(t) = O(t^{\frac{1}{4}(m-2)}), \quad (t \to \infty \text{ in } \boldsymbol{S}_{j,k}). \tag{3.31}$$

### 3.6. *Proof in the general case: conclusion*

Lemma. *If $v$ and $\zeta$ are points of $\mathscr{Q}_\varpi$ in the order $\alpha_j$, $v$, $\zeta$, $\beta_k$, then*

$$|K(\zeta, v)| \leqslant \overline{\lambda}_{j,k}^{-1} \mathrm{e}^{2u\varpi} \overline{E}_{j,k}(u^{2/m}\zeta)\, \overline{E}_{j,k}^{-1}(u^{2/m}v)\, \overline{M}_{j,k}(u^{2/m}\zeta)\, \overline{M}_{j,k}(u^{2/m}v), \tag{3.32}$$

$$|\partial K(\zeta, v)/\partial\zeta| \leqslant \overline{\lambda}_{j,k}^{-1} u^{2/m} \mathrm{e}^{2u\varpi} \overline{E}_{j,k}(u^{2/m}\zeta)\, \overline{E}_{j,k}^{-1}(u^{2/m}v)\, \overline{N}_{j,k}(u^{2/m}\zeta)\, \overline{M}_{j,k}(u^{2/m}v), \tag{3.33}$$

*where*

$$\overline{\lambda}_{j,k} = \min_{p \neq q} |\lambda_{p,q}| \quad (p \in \{j, k\}, \quad q \in \{j, k\}). \tag{3.34}$$

To prove this result, we note first that the constant $\overline{\lambda}_{j,k}$ is non-zero. This follows from condition (i) of §3.3 and the definitions (2.17) and (2.18) by observing that $0 < |q-p| \leqslant |k-j|$. Next, we observe that details of the proof need be supplied only when $k > j$: the analysis for $k < j$ is exactly similar. We denote the part of $\mathscr{Q}_\varpi$ that coincides with the circular arc of radius $\varpi^{2/m}$ by $\mathscr{I}_\varpi$. There are five cases to consider.

(i) Suppose that $\zeta \in \boldsymbol{S}_j$. Then $v \in \boldsymbol{S}_j$. The inequality (3.23) applies unless $\zeta \in \mathscr{I}_\varpi \cap \boldsymbol{S}_j$. However, from the definition of $E_{j,k}$ (§2.5) it is easily verified that the weaker inequality

$$E_{j,k}(u^{2/m}v) \leqslant \mathrm{e}^{u\varpi} E_{j,k}(u^{2/m}\zeta) \tag{3.35}$$

applies without restriction. Substituting in (3.22) by means of (2.29) and using (3.35), we find that

$$|K(\zeta, v)| \leqslant |\lambda_{j,k}^{-1}|\, \mathrm{e}^{2u\varpi} E_{j,k}(u^{2/m}\zeta)\, E_{j,k}^{-1}(u^{2/m}v)\, M_{j,k}(u^{2/m}\zeta)\, M_{j,k}(u^{2/m}v); \tag{3.36}$$

compare (3.24).

(ii) Suppose that $\zeta \in \boldsymbol{S}_q$, where $j+1 \leqslant q \leqslant k$, and $v \in \boldsymbol{S}_j$. Then from (3.29) with $p = j$ we have

$$|K(\zeta, v)| \leqslant \overline{\lambda}_{j,k}^{-1} |U_j(u^{2/m}\zeta)\, U_q(u^{2/m}v) - U_q(u^{2/m}\zeta)\, U_j(u^{2/m}v)|.$$

Referring to (2.29) and (2.27) and recalling that $e(t) \geqslant 1$ for all $t$, we derive

$$|K(\zeta, v)| \leqslant \overline{\lambda}_{j,k}^{-1} e(u^{2/m}\zeta)\, e(u^{2/m}v)\, M_{j,q}(u^{2/m}\zeta)\, M_{j,q}(u^{2/m}v).$$

(iii) Suppose that $\zeta \in \boldsymbol{S}_q$ and $v \in \boldsymbol{S}_p$, where $j+1 \leqslant p \leqslant q-1$ and $j+2 \leqslant q \leqslant k$. Then $v \in \mathscr{I}_\varpi$. From (3.29) we derive

$$|K(\zeta, v)| \leqslant \overline{\lambda}_{j,k}^{-1} |U_p(u^{2/m}\zeta)\, U_q(u^{2/m}v) - U_q(u^{2/m}\zeta)\, U_p(u^{2/m}v)|.$$

Since $|v| = \varpi^{2/m}$ it follows that

$$e(u^{2/m}v) \leqslant e^{2u\varpi} e^{-1}(u^{2/m}v),$$

and hence that

$$|K(\zeta, v)| \leqslant \overline{\lambda}_{j,k}^{-1} e^{2u\varpi} e(u^{2/m}\zeta)\, e^{-1}(u^{2/m}v)\, M_{p,q}(u^{2/m}\zeta)\, M_{p,q}(u^{2/m}v).$$

(iv) Suppose that $\zeta \in \boldsymbol{S}_q$ and $v \in \boldsymbol{S}_q$, where $j+1 \leqslant q \leqslant k-1$. Then we use (3.29) with $p$ replaced by $j$. Using reasoning similar to that of (iii), we arrive at

$$|K(\zeta, v)| \leqslant \overline{\lambda}_{j,k}^{-1} e^{2u\varpi} e(u^{2/m}\zeta)\, e^{-1}(u^{2/m}v)\, M_{j,q}(u^{2/m}\zeta)\, M_{j,q}(u^{2/m}v).$$

(v) Lastly, suppose that $\zeta \in \boldsymbol{S}_k$ and $v \in \boldsymbol{S}_k$. Then (3.35) holds and from (3.22) we deduce that (3.36) again applies.

All possible combinations have now been covered, and by combining the results and referring to the definitions and properties of $E_{j,k}$, $M_{j,k}$, $\overline{E}_{j,k}$ and $\overline{M}_{j,k}$ given in §§2.5 and 3.5, we perceive that (3.32) applies in all cases. The proof of (3.33) is similar.

Having established the bounds (3.32) and (3.33), we may solve the integral equation (3.21) along $\mathscr{Q}_\varpi$ by application of theorem 10.2 of Olver (1974, ch. 6). Corresponding to (3.26) we find that

$$\frac{|\epsilon(u,\zeta)|}{\overline{E}_{j,k}(u^{2/m}\zeta)\, \overline{M}_{j,k}(u^{2/m}\zeta)}, \quad \frac{|\partial\epsilon(u,\zeta)/\partial\zeta|}{u^{2/m}\overline{E}_{j,k}(u^{2/m}\zeta)\, \overline{N}_{j,k}(u^{2/m}\zeta)} \leqslant \frac{\overline{\sigma}_{j,k}}{\overline{\rho}_{j,k}}\left[\exp\left\{\frac{e^{2u\varpi}\overline{\rho}_{j,k}}{m\overline{\lambda}_{j,k}u^{2/m}}\int_{\alpha_j}^{\zeta}\frac{|\phi(u,v)|}{\Omega(u^{2/m}v)}|\mathrm{d}v|\right\}-1\right], \tag{3.37}$$

where the integral is evaluated along $\mathscr{Q}_\varpi$,

$$\overline{\rho}_{j,k} = \sup_{t \in \boldsymbol{S}_{j,k}} \{\Omega(t)\, \overline{M}^2_{j,k}(t)\}, \tag{3.38}$$

and
$$\overline{\sigma}_{j,k} = \sup_{v \in \mathscr{Q}_\varpi} \{\Omega(u^{2/m}v)\, \overline{E}_{j,k}^{-1}(u^{2/m}v)\, \overline{M}_{j,k}(u^{2/m}v)\, |AU_j(u^{2/m}v) + BU_k(u^{2/m}v)|\}. \tag{3.39}$$

In consequence of (3.7) and the first of (3.31) $\overline{\rho}_{j,k}$ is always finite. Also, $\overline{\sigma}_{j,k}$ is finite whenever $\sigma_{j,k}$ is finite; compare remark (*b*) of §3.3. Moreover, because $\epsilon(u,\zeta)$ is twice continuously differentiable along $\mathscr{Q}_\varpi$, it satisfies the differential equation (3.20) everywhere on $\mathscr{Q}_\varpi$.

Let us now suppose that $\zeta$ is any prescribed point of $\mathscr{Q} \cap \boldsymbol{S}_k$, other than the origin, and consider the behaviour of the integral

$$\int_{\mathscr{I}_\varpi} K(\zeta, v)\, \phi(u,v)\, \{AU_j(u^{2/m}v) + BU_k(u^{2/m}v) + \epsilon(u,v)\}\, \mathrm{d}v \tag{3.40}$$

as $\varpi \to 0$. First, since each of the functions appearing within the braces on the right hand side of (3.39) is a continuous function of $v$ at the origin, it follows that $\overline{\sigma}_{j,k}$ is bounded as $\varpi \to 0$. Next, from (3.2), (3.5) and (3.17) we deduce that

$$\phi(u,v) = O(v^{\gamma_1 - 1}) \quad (v \to 0),$$

where $\gamma_1 = \min(\gamma, 1)$. Substituting in (3.37) by means of these results, we conclude that $|\epsilon(u,v)|$ is bounded as $\varpi \to 0$, the bound being uniform with respect to $v \in \mathscr{I}_\varpi$. Lastly, since we also know that $K(\zeta, v)$ is a continuous function of $v$ we deduce that the integral (3.40) vanishes as $\varpi \to 0$.

We have therefore established that when $\zeta \in \mathscr{Q} \cap \boldsymbol{S}_k$ and $\zeta \neq 0$, the solution $\epsilon(u,\zeta)$ of the differential equation (3.20) obtained by analytic continuation along $\mathscr{I}_\varpi$ from $\boldsymbol{S}_j$ to $\boldsymbol{S}_k$ also satisfies the integral equation (3.21) along $\mathscr{Q}$. Furthermore, $\epsilon(u,\zeta)$ is continuous on $\mathscr{Q}$, including the origin. Now we can also construct a solution of (3.21) on $\mathscr{Q}$ by use of (3.24) and direct application of the method of successive approximations.† Furthermore, this method also shows that: (i) there is only one solution that is continuous on $\mathscr{Q}$ and has the property

$$\epsilon(u,\zeta) = \bar{E}_{j,k}(u^{2/m}\zeta)\,\bar{M}_{j,k}(u^{2/m}\zeta)\,O(1)$$

as $\zeta \to \alpha_j$ and also as $\zeta \to \beta_k$; (ii) this solution is subject to the bounds (3.26).

The remainder of the proof is completed as in §3.4.

### 3.7. *Additional remark on theorem* 3.1

It might be thought that the condition on $k$ imposed by §3.3 (i) is a consequence of the method of proof that we have adopted. This is not so, however, and the underlying reason is as follows. Let $m = m_1/m_2$, where $m_1$ and $m_2$ are mutually prime and $m_1 > 1$. Suppose also that $B = 0$ and $k-j = \pm m_1$. Then $\lambda_{j,k} = 0$, and from (2.23) and (2.24) we see that the approximant $f^{-\frac{1}{4}}(u,z)\,AU_j(u^{2/m}\zeta)$ in (3.12) is recessive as $\zeta \to \infty$ in $\boldsymbol{S}_k$, in contrast to its dominant behaviour in all sectors that intervene between $\boldsymbol{S}_j$ and $\boldsymbol{S}_k$. We cannot expect the true solution $w(u,z)$ to be recessive in exactly the same circumstances.

## 4. Asymptotic estimates of the error terms

### 4.1. *Preliminary remarks*

The inequalities for the error term and its derivative supplied by theorem 3.1 are valid for *any* value of the positive parameter $u$. In §1 it was stated that the object of the present paper is to construct uniform asymptotic solutions of the differential equation (1.2) as $u \to \infty$. Accordingly, we now consider circumstances in which the error term is sufficiently small for the theorem to supply meaningful approximations when $u$ is large. On referring to (2.29), (2.30) and (3.12) we see that in the archetypal case, $A = 1$ and $B = 0$, we require

$$\epsilon(u,\zeta) = E_{j,k}(u^{2/m}\zeta)\,M_{j,k}(u^{2/m}\zeta)\,o(1), \quad \partial\epsilon(u,\zeta)/\partial\zeta = E_{j,k}(u^{2/m}\zeta)\,N_{j,k}(u^{2/m}\zeta)\,o(u^{2/m}),$$

as $u \to \infty$, uniformly with respect to $z$. Again, when $A = 1$ and $B = 0$ it is seen from (2.29), (3.9) and (3.14) that $\sigma_{j,k} \leqslant \rho_{j,k}$. Hence from (3.13) both asymptotic conditions just given are met when

$$\mathscr{V}_{\alpha_j,\alpha_k}(H) = o(u^{2/m}) \quad (u \to \infty), \tag{4.1}$$

this variation being taken along the progressive path $\mathscr{P}$.

Whether the condition (4.1) is satisfied depends largely on the manner in which the parameter $u$ enters the functions $f(u,z)$ and $g(u,z)$, and each application may be examined on its merits. In practice, a commonly occurring case is that in which $f(u,z)$ and $g(u,z)$ are independent of $u$. Some simplifications then become available; for example, the principal regions are independent of $u$. In the present section we shall examine this case in detail, using analysis similar to that for real variables given in §5 of Olver (1977*a*).

Before we begin, we introduce the L.G. error-control function defined by

$$F(u,z) = \int \left\{ \frac{1}{f^{\frac{1}{4}}(u,z)} \frac{d^2}{dz^2} \left( \frac{1}{f^{\frac{1}{4}}(u,z)} \right) - \frac{g(u,z)}{f^{\frac{1}{2}}(u,z)} \right\} dz. \tag{4.2}$$

For this integral we shall not use paths that intersect $c$. Any branches may be adopted for $f^{\frac{1}{4}}(u,z)$ and $f^{\frac{1}{2}}(u,z)$, provided that they are continuous and the latter is the square of the former.

† But not, in this instance, by use of theorem 10.2 of Olver (1974, ch. 6), because not all the conditions are satisfied. However, the general method of successive approximations is still applicable, and yields the stated results.



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Next, we give the following expression for the error-control function $H(u,z)$, obtained by substituting (3.6) in (3.10):

$$H(u,z) = \frac{m}{2}\int\left\{\frac{1}{f^{\frac{1}{4}}(u,z)}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\left(\frac{1}{f^{\frac{1}{4}}(u,z)}\right) - \frac{g(u,z)}{f^{\frac{1}{2}}(u,z)}\right\}\frac{\zeta^{\frac{1}{2}(m-2)}\,\mathrm{d}z}{\Omega(u^{2/m}\zeta)} - \frac{m^2-4}{16}\int\frac{\mathrm{d}\zeta}{\zeta^2\Omega(u^{2/m}\zeta)}. \tag{4.3}$$

The two integrals in this expression may not converge separately at $z = c$, but this is immaterial.

### 4.2. *Asymptotic estimates for paths*

THEOREM 4.1. (i) *Assume the conditions of §§3.1 and 3.2 are satisfied, and that the functions $f$ and $g$ are independent of $u$. Let $a_j$ and $a_k$ be arbitrary points in the closures of $\mathbf{D}_j$ and $\mathbf{D}_k$, respectively, including the point at infinity, and $\mathscr{P}$ be a progressive path connecting $a_j$ and $a_k$. Assume also that each of the functions $1/\xi(z)$ and $F(z)$, defined by (3.11) and (4.2), is of bounded variation as $z \to a_j$ or $a_k$ along $\mathscr{P}$. Then*

$$u^{-2/m}\mathscr{V}_{\mathscr{P}}(H) = O(\psi_m) \quad (u\to\infty), \tag{4.4}$$

*where*

$$\left.\begin{aligned}\psi_m &= u^{-1} \quad (0<m<2(1+\gamma_1)),\\ \psi_m &= u^{-1}\ln u \quad (m = 2(1+\gamma_1)),\\ \psi_m &= u^{-2(1+\gamma_1)/m} \quad (m>2(1+\gamma_1)),\end{aligned}\right\} \tag{4.5}$$

*and*

$$\gamma_1 = \min(\gamma, 1). \tag{4.6}$$

(ii) *If, in addition, $g(z)$ is analytic at $c$, then*

$$u^{-2/m}\mathscr{V}_{\mathscr{P}}(H) = O(\chi_m) \quad (u\to\infty), \tag{4.7}$$

*where*† $\chi_m = u^{-1}\ (0<m<4),\quad \chi_m = u^{-1}\ln u\ (m=4),\quad \chi_m = u^{-4/m}\ (m>4).$ (4.8)

The proof is given in the following two subsections.

### 4.3. *Proof of theorem 4.1 when $a_j$, $a_k$, $\zeta(a_j)$ and $\zeta(a_k)$ are finite*

In this subsection we assume that $a_j$ and $a_k$ are finite and $\zeta$ remains finite as $z\to a_j$ or $a_k$ along $\mathscr{P}$. We also assume that $c\in\mathscr{P}$. (Compare remark (*a*) in §3.3.)

Let $\alpha_j$, $\alpha_k$ and $\mathscr{Q}$ again denote the $\zeta$-maps of $a_j$, $a_k$ and $\mathscr{P}$, respectively. Referring to (3.10) and transforming to the $\zeta$-plane, we find that

$$\mathscr{V}_{\mathscr{P}}(H) = \int_{\alpha_j}^{\alpha_k}\left|\left\{\frac{1}{f^{\frac{1}{4}}(z)}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\left(\frac{1}{f^{\frac{1}{4}}(z)}\right) - \frac{g(z)}{f^{\frac{1}{2}}(z)}\right\}\frac{\mathrm{d}z}{\mathrm{d}\zeta}\right|\frac{|\mathrm{d}\zeta|}{\Omega(u^{2/m}\zeta)}, \tag{4.9}$$

where the integral is evaluated along $\mathscr{Q}$. Because $\mathrm{d}z/\mathrm{d}\zeta$, $f^{-\frac{1}{4}}(z)$ and the derivatives of $f^{-\frac{1}{4}}(z)$ are continuous on $\mathscr{P}$, it follows that

$$\left|\frac{1}{f^{\frac{1}{4}}(z)}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\left(\frac{1}{f^{\frac{1}{4}}(z)}\right)\frac{\mathrm{d}z}{\mathrm{d}\zeta}\right| \leqslant A \quad (\zeta\in\mathscr{Q}),$$

where $A$ is an assignable finite constant. Hence on making the choice (3.8) for $\Omega$ we have

$$\int_{\alpha_j}^{\alpha_k}\left|\frac{1}{f^{\frac{1}{4}}(z)}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\left(\frac{1}{f^{\frac{1}{4}}(z)}\right)\frac{\mathrm{d}z}{\mathrm{d}\zeta}\right|\frac{|\mathrm{d}\zeta|}{\Omega(u^{2/m}\zeta)} \leqslant A\int_{\alpha_j}^{\alpha_k}\frac{1+u^{2/m}|\zeta|}{1+u|\zeta|^{\frac{1}{2}m}}|\mathrm{d}\zeta|.$$

† This notation differs slightly from that of Olver (1977*b*). Also, when $g(z)$ is analytic at $c$ we have $\gamma_1 = 1$, and the estimates (4.4) and (4.7) are the same.

Denote the arc parameter of $\mathscr{Q}$, measured from $\zeta = 0$, by $\tau$ and let $\tau_j(<0)$ and $\tau_k(>0)$ correspond to $\alpha_j$ and $\alpha_k$, respectively. Then on $\mathscr{Q}$

$$1 \leqslant |\tau/\zeta| \leqslant A,$$

where the symbol $A$ is now being used generically. Since $|\mathrm{d}\zeta/\mathrm{d}\tau| = 1$, it follows that

$$\frac{1}{u^{2/m}}\int_{\alpha_j}^{\alpha_k}\left|\frac{1}{f^{\frac{1}{4}}(z)}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\left(\frac{1}{f^{\frac{1}{4}}(z)}\right)\frac{\mathrm{d}z}{\mathrm{d}\zeta}\right|\frac{|\mathrm{d}\zeta|}{\Omega(u^{2/m}\zeta)} \leqslant \frac{A}{u^{2/m}}\int_{\tau_j}^{\tau_k}\frac{1+u^{2/m}|\tau|}{1+u|\tau|^{\frac{1}{2}m}}\mathrm{d}\tau. \tag{4.10}$$

On transforming to the integration variable $v \equiv \pm u^{2/m}\tau$, we obtain

$$\frac{1}{u^{2/m}}\int_{\tau_j}^{\tau_k}\frac{1+u^{2/m}|\tau|}{1+u|\tau|^{\frac{1}{2}m}}\mathrm{d}\tau = \frac{1}{u^{4/m}}\left(\int_0^{u^{2/m}\tau_k}+\int_0^{u^{2/m}|\tau_j|}\right)\frac{1+v}{1+v^{\frac{1}{2}m}}\mathrm{d}v = O(\chi_m), \tag{4.11}$$

as $u \to \infty$, where $\chi_m$ is defined by (4.8).

Next, if $g(z)$ is analytic at $c$, then $|g(z)f^{-\frac{1}{2}}(z)|$ is bounded on $\mathscr{P}$. By similar analysis we derive

$$\frac{1}{u^{2/m}}\int_{\alpha_j}^{\alpha_k}\left|\frac{g(z)}{f^{\frac{1}{2}}(z)}\frac{\mathrm{d}z}{\mathrm{d}\zeta}\right|\frac{|\mathrm{d}\zeta|}{\Omega(u^{2/m}\zeta)} = O(\chi_m). \tag{4.12}$$

Alternatively, assume that $g(z)$ is singular at $c$. From (3.2) we have

$$|g(z)| \leqslant A|z-c|^{\gamma-1} \quad (z \in \mathscr{P}),$$

and hence

$$|g(z)| \leqslant A|\zeta|^{\gamma-1} \quad (\zeta \in \mathscr{Q}).$$

Using this inequality we find that

$$\frac{1}{u^{2/m}}\int_{\alpha_j}^{\alpha_k}\left|\frac{g(z)}{f^{\frac{1}{2}}(z)}\frac{\mathrm{d}z}{\mathrm{d}\zeta}\right|\frac{|\mathrm{d}\zeta|}{\Omega(u^{2/m}\zeta)} \leqslant \frac{A}{u^{2(1+\gamma)/m}}\left(\int_0^{u^{2/m}\tau_k}+\int_0^{u^{2/m}|\tau_j|}\right)\frac{1+v}{1+v^{\frac{1}{2}m}}v^{\gamma-1}\,\mathrm{d}v; \tag{4.13}$$

compare (4.11). For large $u$ the right hand side is

$$O(u^{-1}) \quad (0 < m < 2(1+\gamma)), \quad O(u^{-1}\ln u) \quad (m = 2(1+\gamma)), \quad O(u^{-2(1+\gamma)/m}) \quad (m > 2(1+\gamma)). \tag{4.14}$$

To complete the proof of part (i) of theorem 4.1 in the present circumstances, we combine (4.10), (4.11), (4.13) and (4.14), and absorb all the error terms in the estimate $O(\psi_m)$, where $\psi_m$ is given by (4.5) and (4.6). To prove part (ii) we combine (4.10), (4.11) and (4.12).

### 4.4. *Conclusion of the proof of theorem* 4.1

Suppose now that $c \in \mathscr{P}$ and either $a_j$ is at infinity, or $|\zeta| \to \infty$ as $z \to a_j$ along $\mathscr{P}$. We introduce an arbitrary fixed point $\hat{a}_j$ on $\mathscr{P}$ between $a_j$ and $c$. The analysis of §4.3 shows that $u^{-2/m}\mathscr{V}_{\hat{a}_j, a_k}(H)$ is estimated by $O(\psi_m)$ or $O(\chi_m)$, according as the hypotheses of (i) or (ii) apply.

To estimate the contribution from the part of $\mathscr{P}$ lying between $a_j$ and $\hat{a}_j$, we use the expression (4.3) for $H$, again with the choice (3.8) for $\Omega$. Since $|\zeta|$ is bounded away from zero on the part of $\mathscr{P}$ under consideration, we have

$$\frac{|\zeta|^{\frac{1}{2}(m-2)}}{\Omega(u^{2/m}\zeta)} = \frac{(u^{2/m}|\zeta|)^{\frac{1}{2}(m-2)}}{u^{(m-2)/m}}\,\frac{1+u^{2/m}|\zeta|}{1+(u^{2/m}|\zeta|)^{\frac{1}{2}m}} \leqslant \frac{A}{u^{(m-2)/m}},$$

provided that $u$, also, is bounded away from zero. Hence

$$\frac{m}{2}\int_{a_j}^{\hat{a}_j}\left|\left\{\frac{1}{f^{\frac{1}{4}}(z)}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\left(\frac{1}{f^{\frac{1}{4}}(z)}\right)-\frac{g(z)}{f^{\frac{1}{2}}(z)}\right\}\frac{\zeta^{\frac{1}{2}(m-2)}\mathrm{d}z}{\Omega(u^{2/m}\zeta)}\right| \leqslant \frac{A}{u^{(m-2)/m}}\mathscr{V}_{a_j,\hat{a}_j}(F), \tag{4.15}$$

the last quantity being finite, by hypothesis.

For the other term in (4.3), let $\hat{\alpha}_j$ be the $\zeta$-map of $\hat{a}_j$. Since $|\zeta|$ is bounded away from zero, we may choose $u$ so that $u^{2/m}|\zeta| \geqslant 1$ everywhere. Then

$$\int_{\alpha_j}^{\hat{\alpha}_j} \left| \frac{d\zeta}{\zeta^2 \Omega(u^{2/m}\zeta)} \right| = \int_{\alpha_j}^{\hat{\alpha}_j} \frac{1+u^{2/m}|\zeta|}{1+u|\zeta|^{\frac{1}{2}m}} \left| \frac{d\zeta}{\zeta^2} \right| \leqslant \int_{\alpha_j}^{\hat{\alpha}_j} \frac{2u^{2/m}|\zeta|}{u|\zeta|^{\frac{1}{2}m}} \left| \frac{d\zeta}{\zeta^2} \right| = \frac{4}{m} u^{(2-m)/m} \mathscr{V}_{\alpha_j, \hat{\alpha}_j}(\zeta^{-\frac{1}{2}m}). \quad (4.16)$$

Since $\zeta^{-\frac{1}{2}m}$ is a multiple of $1/\xi(z)$ the last expression is finite, by hypothesis. On combining this bound with (4.15) we conclude that

$$u^{-2/m}\mathscr{V}_{a_j, \hat{a}_j}(H) = O(u^{-1}) \quad (u \to \infty).$$

This estimate is absorbable in $O(\psi_m)$ or $O(\chi_m)$; in consequence (4.4) and (4.7) again apply.

A similar proof holds when $a_k$ is at infinity or $|\zeta| \to \infty$ as $z \to a_k$ along $\mathscr{P}$.

Lastly, suppose that $c \notin \mathscr{P}$. Then we use the same proof as in the present subsection, except that $\hat{a}_j$ now denotes a point on the common boundary of $\mathbf{D}_j$ and $\mathbf{D}_k$. Clearly in this case we have the stronger result

$$u^{-2/m}\mathscr{V}_{\mathscr{P}}(H) = O(u^{-1}) \quad (u \to \infty),$$

valid for all values of $m$.

This completes the proof of theorem 4.1.

### 4.5. *Asymptotic estimates for domains*

Sometimes we need an estimate for the error term of theorem 3.1 that is uniform for a region rather than a single path. A typical result of this nature is as follows:

THEOREM 4.2. *Assume the conditions of* §§3.1 *and* 3.2 *are satisfied, and that the functions f and g are independent of u. Let $a_k$ be a boundary point of $\mathbf{D}_k$ (possibly at infinity) such that $|\xi(a_k)| = \infty$, and $\theta_1$ and $\theta_2$ be constants such that $-\frac{1}{2}\pi \leqslant \theta_1 < \theta_2 \leqslant \frac{1}{2}\pi$. Assume also that for every value of $\theta$ in the closed interval $[\theta_1, \theta_2]$ there is a path $\mathscr{P}_\theta$ that has the equation*

$$\operatorname{ph} \xi(z) = \theta \quad (4.17)$$

*and lies in $\mathbf{D}_k$. If*

$$\mathscr{V}_{z, a_k}(F) \to 0 \quad (z \to a_k), \quad (4.18)$$

*where the variation is evaluated along $\mathscr{P}_\theta$, then*

$$u^{-2/m}\mathscr{V}_{c, a_k}(H) = O(\psi_m) \quad (u \to \infty), \quad (4.19)$$

*again with the variation evaluated along $\mathscr{P}_\theta$. Furthermore, if* (4.18) *applies uniformly with respect to $\theta$, then so does* (4.19).

It will be observed that the condition (4.17) ensures that $\mathscr{P}_\theta$ is a progressive path. The proof of theorem 4.2 is a straightforward extension of that of theorem 4.1, and it is unnecessary to include details. The condition of uniform convergence, as applied to (4.18), could be eased to bounded convergence (see, for example, Apostol 1957, p. 405), but this is unlikely to be important in applications.

# PART B. CONNECTION FORMULAE

## 5. MAIN CONNECTION THEOREM

### 5.1. *Preliminary definitions*

Throughout the present section we suppose that the conditions of §3.1 are satisfied. Before stating the main result we reformulate the geometric concepts introduced in §§3.2 and 3.3 in terms of a new variable $\xi_c(u, z)$, defined below, in order to avoid explicit reference to the variables $\zeta(u, z)$ and $\xi(u, z)$ in applications.

The curves in the $z$-plane having the equation

$$\mathrm{Re}\int_c^z f^{\frac{1}{2}}(u,t)\,\mathrm{d}t = 0 \tag{5.1}$$

are called the *principal curves associated with the transition point c*. Either branch of $f^{\frac{1}{2}}(u,t)$ may be used in this definition, provided that continuity is maintained. When $m = 2, 3, 4, \ldots$, there are $m$ distinct principal curves and they occupy the same Riemann sheet. For other values of $m$ the principal curves lie on two or more Riemann sheets. In all cases, however, adjacent principal curves intersect at $c$ at an angle $2\pi/m$. The theory of conformal mapping shows that a principal curve can terminate only at $c$ or a boundary point of $\boldsymbol{D}$, and also that no principal curve can intersect itself or any other principal curve on the same Riemann sheet, except at $c$.

The principal curves are evidently boundaries of the principal regions introduced in §3.2, and we note that each principal region includes its bounding principal curves. *We shall suppose that in each principal region there is a one-to-one relation between z and any continuous branch of the integral* $\int f^{\frac{1}{2}}(u,z)\,\mathrm{d}z$. This is equivalent to the assumption of §3.2 that the mapping from $\boldsymbol{D}$ on to $\boldsymbol{\Delta}$ is one-to-one.

Starting with an arbitrarily designated principal region $\boldsymbol{D}_0$, we define $\boldsymbol{D}_1$, $\boldsymbol{D}_2$, ..., to be the successive principal regions that are encountered as we pass around $c$ in the positive rotational sense; similarly $\boldsymbol{D}_{-1}$, $\boldsymbol{D}_{-2}$, ..., denote the successive principal regions encountered in the negative rotational sense. Whatever choice is adopted for $\boldsymbol{D}_0$, the labelling can be made consistent with that of §3.2 by appropriate choice of branch of $f_0^{1/m}$ in the expansion (3.5). We shall refer to $\boldsymbol{D}_j \cap \boldsymbol{D}_{j+1}$ and $\boldsymbol{D}_j \cap \boldsymbol{D}_{j-1}$ as the *left* and *right boundaries*, respectively, of $\boldsymbol{D}_j$; compare §2.4.

Next, in each principal region $\boldsymbol{D}_j$ we define

$$\xi_c(u,z) = \int_c^z f^{\frac{1}{2}}(u,t)\,\mathrm{d}t, \tag{5.2}$$

*taking the branch that is continuous and has non-negative real part*. This determines a function of $z$ that is continuous throughout $\boldsymbol{D}$, except on the principal curves. Clearly $\xi_c(u,z)$ is positive imaginary on the left boundary of $\boldsymbol{D}_j$ and negative imaginary on the right boundary of $\boldsymbol{D}_j$. Since the left boundary of $\boldsymbol{D}_j$ is also the right boundary of $\boldsymbol{D}_{j+1}$, and the right boundary of $\boldsymbol{D}_j$ is also the left boundary of $\boldsymbol{D}_{j-1}$, it is clear that $\xi_c(u,z)$ changes sign on crossing a principal curve.

In addition to the left and right boundaries, we also need to define the *left*, *right* and *internal parts* of $\boldsymbol{D}_j$. These are the point sets in $\boldsymbol{D}_j$ that satisfy

$$0 \leqslant \mathrm{ph}\,\xi_c(u,z) \leqslant \tfrac{1}{2}\pi, \quad -\tfrac{1}{2}\pi \leqslant \mathrm{ph}\,\xi_c(u,z) \leqslant 0, \quad |\mathrm{ph}\,\xi_c(u,z)| \leqslant \tfrac{1}{2}\delta\pi, \tag{5.3}$$

respectively, where $\delta$ again denotes an arbitrary positive constant less than unity; compare again §2.4.

The last concept is that of a *progressive path*. This is any Jordan arc $\mathscr{P}$ comprising a finite chain of $R_2$ arcs, in the sense of Olver (1974, p. 147), and having the property that $\mathrm{Re}\,\xi_c(u,z)$ is monotonic on the intersection of $\mathscr{P}$ with any principal region. It needs to be clarified that this does not require $\mathrm{Re}\,\xi_c(u,z)$ to be monotonic along the whole of $\mathscr{P}$; $\mathrm{Re}\,\xi_c(u,z)$ may be non-increasing in one principal region and non-decreasing in another.

It is easily verified that the definition just given is consistent with that of §3.3. Thus $\mathscr{P}$ can lie in at most two principal regions $\boldsymbol{D}_j$ and $\boldsymbol{D}_k$, say. Furthermore, unless $k-j = \pm 1$, $\mathscr{P}$ has to pass through $c$ in order to enter one principal region from the other. Typical progressive paths are indicated in figure 5.

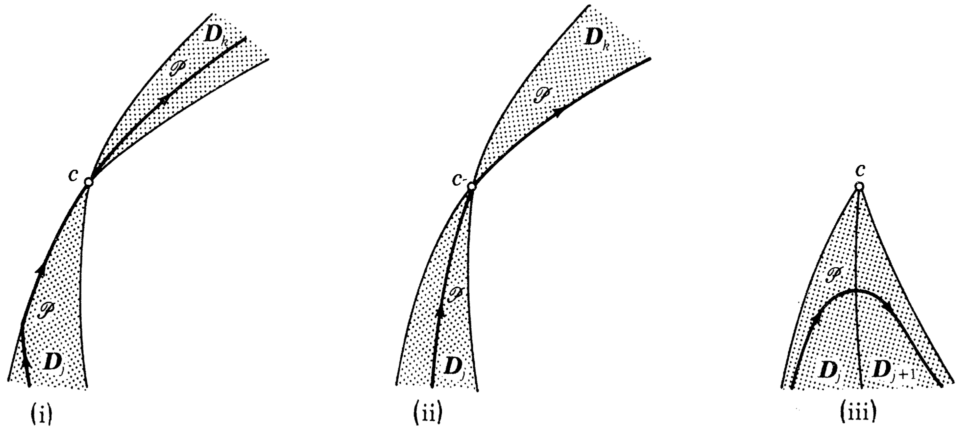


FIGURE 5. $z$-plane: progressive paths $\mathscr{P}$.

### 5.2. *Statement of theorem*

In this subsection we assume the conditions and notation of §5.1, and also that $j$ and $k$ are integers satisfying condition (i) of §3.3. We suppose that $a_j$ and $a_k$ are given points (other than $c$) in the closures of $\boldsymbol{D}_j$ and $\boldsymbol{D}_k$, respectively, including boundary points or points at infinity, and $\mathscr{P}$ is a progressive path connecting $a_j$ and $a_k$. We also suppose that $\hat{\delta}$ denotes a given number that is independent of $z$ (but may depend on $u$) and satisfies $0 \leqslant \hat{\delta} < 1$. Before stating the main theorem in this section we introduce the following notations.

First,

$$\Delta_j(u) = (1+\hat{\delta})\,[1+\Theta\{u\xi_c(u,a_j)\}]-1, \tag{5.4}$$

where $\Theta(t)$ is defined by (2.9).

Secondly,

$$X_{j,k}(u) = \Delta_j(u)\,\{1+T_j|\lambda_{j,k}^{-1}\,\lambda_{j,k\mp1}\,\mathrm{e}^{-2u\xi_c(u,a_j)}|\}, \tag{5.5}$$

where the upper or lower sign is taken according as $a_j$ lies in the left or right part of $\boldsymbol{D}_j$, $\lambda_{j,k}$ is defined by (2.17) and (2.18), and $T_j = 1$ or $0$ according as $|\xi_c(u,a_j)|$ is finite or infinite. Thus when $|\xi_c(u,a_j)| = \infty$ we have $\Theta\{u\xi_c(u,a_j)\} = 0$ and $X_{j,k}(u) = \Delta_j(u) = \hat{\delta}$.

Lastly, when $z \in \mathscr{P}$

$$\Lambda_{j,k}(u,z) = C_{j,k}\{1+X_{j,k}(u)+T_j|\lambda_{j,k}^{-1}|\Delta_j(u)\}[\exp\{m^{-1}|\lambda_{j,k}^{-1}|\rho_{j,k}\,u^{-2/m}\mathscr{V}_{a_j,z}(H)\}-1], \tag{5.6}$$

where $C_{j,k}$, $\rho_{j,k}$ and $H \equiv H(u,z)$ are defined by (2.38), (3.9) and (3.10), and the variation of $H$ is evaluated along $\mathscr{P}$.

THEOREM 5.1. *Let $w(u,z)$ be a solution of* (3.1) *with the properties*

$$f^{\frac{1}{4}}(u,z)\,w(u,z) \sim (1+\eta)\,\mathrm{e}^{-u\xi_c(u,z)}, \tag{5.7}$$

$$\partial\{f^{\frac{1}{4}}(u,z)\,w(u,z)\}/\partial z \sim -(1+\eta^1)\,uf^{\frac{1}{2}}(u,z)\,\mathrm{e}^{-u\xi_c(u,z)}, \tag{5.8}$$

*as $z \to a_j$ along $\mathscr{P}$, where $\eta$ and $\eta^1$ are independent of $z$ and $|\eta|$, $|\eta^1| \leqslant \hat{\delta}$. Assume also that $1/\xi_c(u,z)$ and $F(u,z)$ are each of bounded variation as $z \to a_j$ or $a_k$ along $\mathscr{P}$, where $F(u,z)$ is defined by* (4.2). *Then the* *analytic continuation of* $w(u,z)$ – *obtained by passing around the neighbourhood of* $c$ *from* $\mathbf{D}_j$ *to* $\mathbf{D}_k$ *in the same sense as the sign of* $(k-j)$ – *has the following properties*:

(i) *If* $z \in \mathscr{P} \cap \mathbf{D}_k$, *then*

$$f^{\frac{1}{4}}(u,z)\,w(u,z) = \mathrm{i}^{k-j-1}\{\lambda_{j,k}+\epsilon_{\pm1}(u,z)\}\,\mathrm{e}^{u\xi_c(u,z)} \pm \mathrm{i}^{k-j}\{\lambda_{j,k\pm1}+\epsilon_{\pm2}(u,z)\}\,\mathrm{e}^{-u\xi_c(u,z)}, \quad (5.9)$$

$$\partial\{f^{\frac{1}{4}}(u,z)\,w(u,z)\}/\partial z = \mathrm{i}^{k-j-1}\{\lambda_{j,k}+\epsilon^1_{\pm1}(u,z)\}\,uf^{\frac{1}{2}}(u,z)\,\mathrm{e}^{u\xi_c(u,z)} \mp \mathrm{i}^{k-j}\{\lambda_{j,k\pm1}+\epsilon^1_{\pm2}(u,z)\}\,uf^{\frac{1}{2}}(u,z)\,\mathrm{e}^{-u\xi_c(u,z)}, \quad (5.10)$$

*where*

$$|\epsilon_{\pm1}(u,z)|,\quad |\epsilon^1_{\pm1}(u,z)| \leqslant [1+\mathit{\Theta}\{u\xi_c(u,z)\}]\,[\mathit{\Lambda}_{j,k}(u,z)+|\lambda_{j,k}|\{1+X_{j,k}(u)\}]-|\lambda_{j,k}|, \quad (5.11)$$

$$|\epsilon_{\pm2}(u,z)|,\quad |\epsilon^1_{\pm2}(u,z)| \leqslant [1+\mathit{\Theta}\{u\xi_c(u,z)\}]\,[|\lambda_{j,k\pm1}|\{1+X_{j,k}(u)\} + T_j\,\mathit{\Delta}_j(u)\,|\lambda^{-1}_{j,k}\,\mathrm{e}^{-2u\xi_c(u,a_j)}|]-|\lambda_{j,k\pm1}|, \quad (5.12)$$

*the upper or lower signs being taken throughout according as* $z$ *lies in the left or right part of* $\mathbf{D}_k$.

(ii) *If* $\mathrm{Re}\,\xi_c(u,z)\to\infty$ *as* $z\to a_k$ *along* $\mathscr{P}$, *then*

$$f^{\frac{1}{4}}(u,z)\,w(u,z) \sim \mathrm{i}^{k-j-1}(\lambda_{j,k}+\kappa_0)\,\mathrm{e}^{u\xi_c(u,z)}, \quad (5.13)$$

$$\partial\{f^{\frac{1}{4}}(u,z)\,w(u,z)\}/\partial z \sim \mathrm{i}^{k-j-1}(\lambda_{j,k}+\kappa_0)uf^{\frac{1}{2}}(u,z)\,\mathrm{e}^{u\xi_c(u,z)}, \quad (5.14)$$

*as* $z\to a_k$ *along* $\mathscr{P}$, *where* $\kappa_0$ *is independent of* $z$ *and*

$$|\kappa_0| \leqslant \mathit{\Lambda}_{j,k}(u,a_k)+|\lambda_{j,k}|\,X_{j,k}(u). \quad (5.15)$$

(iii) *If* $\mathscr{P}\cap\mathbf{D}_k$ *coincides with the left or right boundary of* $\mathbf{D}_k$ *and* $|\xi_c(u,a_k)| = \infty$, *then*

$$f^{\frac{1}{4}}(u,z)\,w(u,z) = \mathrm{i}^{k-j-1}(\lambda_{j,k}+\kappa_{\pm1})\,\mathrm{e}^{u\xi_c(u,z)} \pm \mathrm{i}^{k-j}(\lambda_{j,k\pm1}+\kappa_{\pm2})\,\mathrm{e}^{-u\xi_c(u,z)}+o(1), \quad (5.16)$$

$$\partial\{f^{\frac{1}{4}}(u,z)\,w(u,z)\}/\partial z = uf^{\frac{1}{2}}(u,z)\,\{\mathrm{i}^{k-j-1}(\lambda_{j,k}+\kappa_{\pm1})\,\mathrm{e}^{u\xi_c(u,z)} \mp \mathrm{i}^{k-j}(\lambda_{j,k\pm1}+\kappa_{\pm2})\,\mathrm{e}^{-u\xi_c(u,z)}+o(1)\}, \quad (5.17)$$

*as* $z\to a_k$ *along* $\mathscr{P}$, *where* $\kappa_{\pm1}$ *and* $\kappa_{\pm2}$ *are independent of* $z$ *and subject to the bounds*

$$|\kappa_{\pm1}| \leqslant \mathit{\Lambda}_{j,k}(u,a_k)+|\lambda_{j,k}|\,X_{j,k}(u), \quad (5.18)$$

$$|\kappa_{\pm2}| \leqslant \mathit{\Lambda}_{j,k}(u,a_k)+|\lambda_{j,k\pm1}|\,X_{j,k}(u)+T_j\,\mathit{\Delta}_j(u)\,|\,\lambda^{-1}_{j,k}\,\mathrm{e}^{-2u\xi_c(u,a_j)}|, \quad (5.19)$$

*and the upper or lower signs are taken throughout according as* $a_k$ *lies on the left or right boundary of* $\mathbf{D}_k$.

*In* (5.9) *to* (5.17) $f^{\frac{1}{4}}(u,z)$ *denotes the branch obtained by analytic continuation from that used in* (5.7) *and* (5.8) *in the same manner as for* $w(u,z)$, *and the branch of* $f^{\frac{1}{2}}(u,z)$ *is the same as that used in* (5.2) *in constructing* $\xi_c(u,z)$.

This theorem is proved in the next three subsections.

*Remarks.* (*a*) If $a_j$ lies in the intersection of the left and right parts of $\mathbf{D}_j$, that is, if $\mathrm{ph}\,\xi_c(u,a_j) = 0$, then either ambiguous sign may be chosen in (5.5). Similarly, when $\mathrm{ph}\,\xi_c(u,z) = 0$ either upper or lower signs may be adopted throughout (5.9) to (5.12).

(*b*) The relations (5.13) and (5.14) are valid whenever the right hand side of (5.15) is less than $|\lambda_{j,k}|$: this condition ensures that the factor $\lambda_{j,k}+\kappa_0$ does not vanish.

### 5.3. *Asymptotic forms of the solutions at boundary points of* $\mathbf{D}$

Theorem 5.1 applies only when there actually exists a solution $w(u,z)$ with the properties (5.7) and (5.8). If $a_j$ is an interior point of $\mathbf{D}$, then $f(u,a_j)$, $1/f(u,a_j)$ and $\xi_c(u,a_j)$ are all finite, and from the theory of linear differential equations we know that $w(u, z)$ exists whatever values are assigned to $\eta$ and $\eta^1$. In other circumstances, however, $w(u, z)$ may exist only when $\eta = \eta^1$. This is illustrated by the following result:

LEMMA 5.1. *Assume the conditions and notation of* § 5.1 *and let* $a_j$ *be a boundary point of* $\mathbf{D}_j$ *or a point at infinity in* $\mathbf{D}_j$. *Assume also that* $|\xi_c(u, z)| \to \infty$ *and* $\mathscr{V}(F)$ *converges as* $z \to a_j$ *along a given progressive path* $\mathscr{P}$ *in* $\mathbf{D}_j$. *Then the differential equation* (3.1) *has a unique solution* $w_j(u, z)$ *that is holomorphic in* $\mathbf{D}$ *and has the property*

$$w_j(u, z) \sim f^{-\frac{1}{4}}(u, z)\, \mathrm{e}^{-u\xi_c(u, z)} \quad (z \to a_j \text{ along } \mathscr{P}). \tag{5.20}$$

*Furthermore,*
$$\partial\{f^{\frac{1}{4}}(u, z)\, w_j(u, z)\}/\partial z \sim -uf^{\frac{1}{2}}(u, z)\, \mathrm{e}^{-u\xi_c(u, z)} \quad (z \to a_j \text{ along } \mathscr{P}). \tag{5.21}$$

The existence of $w_j(u, z)$ is an immediate consequence of theorem 11.1 of Olver (1974, ch. 6). This theorem also shows that $w_j(u, z)$ has the property (5.21). Next, if $\mathrm{Re}\, \xi_c(u, z) \to \infty$ as $z \to a_j$, then $w_j(u, z)$ is a recessive solution and known to be unique.

It therefore remains to show that $w_j(u, z)$ is unique in the case when $\mathrm{Re}\, \xi_c(u, z)$ is bounded as $z \to a_j$. Because $\mathrm{Re}\, \xi_c(u, z)$ is non-decreasing it must tend to a constant. Hence $|\mathrm{Im}\, \xi_c(u, z)| \to \infty$. Another deduction from theorem 11.1 of Olver (1974, ch. 6) is that there is a second solution $\tilde{w}_j(u, z)$, say, such that

$$\tilde{w}_j(u, z) = f^{-\frac{1}{4}}(u, z)\, \mathrm{e}^{u\xi_c(u, z)}\{1 + \tilde{\epsilon}_j(u, z)\}, \tag{5.22}$$

where
$$|\tilde{\epsilon}_j(u, z)| < \tfrac{1}{2}, \quad |u^{-1}f^{-\frac{1}{2}}(u, z)\, \partial\tilde{\epsilon}_j(u, z)/\partial z| < \tfrac{1}{2}, \tag{5.23}$$

for all points on $\mathscr{P}$ that are sufficiently close to $a_j$. On letting $z \to a_j$ in the Wronskian relation for $f^{\frac{1}{4}}(u, z)\, w_j(u, z)$ and $f^{\frac{1}{4}}(u, z)\, \tilde{w}_j(u, z)$ and using (5.20)–(5.23), we see that $w_j(u, z)$ and $\tilde{w}_j(u, z)$ are linearly independent. Also,

$$f^{\frac{1}{4}}(u, z)\, \mathrm{e}^{u\xi_c(u, z)} \tilde{w}_j(u, z) = \mathrm{e}^{2u\xi_c(u, z)}\{1 + \tilde{\epsilon}_j(u, z)\}.$$

In consequence of the first of (5.23) and the facts that $\mathrm{Re}\, \xi_c(u, z) \to$ a constant and $|\mathrm{Im}\, \xi_c(u, z)| \to \infty$, the right hand side of the last relation cannot tend to a constant value as $z \to a_j$ along $\mathscr{P}$. Hence the solution $w_j(u, z)$ determined by (5.20) is unique, and the lemma is established.

## 5.4. *Proof of theorem* 5.1: *determination of A and B*

To prove theorem 5.1 we shall use the uniform approximation supplied by theorem 3.1, and replace the functions $U_j(u^{2/m}\zeta)$ and $U_k(u^{2/m}\zeta)$ by their appropriate asymptotic representations for large arguments. We must first show that the conditions of theorem 3.1 are satisfied; in particular that the error-control function $H(u, z)$ is of bounded variation along $\mathscr{P}$. On any compact segment of $\mathscr{P}$ that does not include $a_j$ or $a_k$ this result follows immediately from the definition (3.10) and the properties of the functions $g(u, z)$, $\hat{f}(u, z)$ and $\Omega(u^{2/m}\zeta)$ given in §§ 3.1 and 3.2. To show $\mathscr{V}(H)$ converges as $z \to a_j$ or $a_k$, we impose an extra condition on the balancing function, given by

$$\Omega(t) \sim \nu |t|^{\frac{1}{2}(m-2)} \tag{5.24}$$

as $t \to \infty$, uniformly with respect to $\mathrm{ph}\, t$, where $\nu$ is a positive constant, and use analysis similar to that of § 4.4. We note, incidentally, that (3.7) is satisfied whenever (5.24) holds.

The remaining conditions of theorem 3.1 are easily verified. Hence the solution $w(u, z)$ satisfying (5.7) and (5.8) can be expressed in the form (3.12). The first problem is to evaluate the coefficients $A$ and $B$. Considerable care will be needed in the choice of branches of the fractional powers that occur in subsequent analysis.





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For brevity, we write temporarily

$$\hat{U}_j(\zeta) = \zeta^{\frac{1}{4}(m-2)} U_j(u^{2/m}\zeta), \quad \hat{U}_k(\zeta) = \zeta^{\frac{1}{4}(m-2)} U_k(u^{2/m}\zeta), \quad \hat{\epsilon}(u,\zeta) = \zeta^{\frac{1}{4}(m-2)} \epsilon(u,\zeta), \tag{5.25}$$

where in each equation the branch of $\zeta^{\frac{1}{4}(m-2)}$ is determined as in § 3, that is,

$$|\zeta|^{\frac{1}{4}(m-2)} \exp\{\mathrm{i}\tfrac{1}{4}(m-2)\,\mathrm{ph}\,\zeta\}$$

with ph $\zeta$ in the interval $[(2j-1)\,\pi/m,\ (2j+1)\,\pi/m]$ when $\zeta \in \boldsymbol{S}_j$. Substituting in (3.12) by means of (3.6), we find that

$$(\tfrac{1}{2}m)^{-\frac{1}{2}} f^{\frac{1}{4}}(u,z)\, w(u,z) = A\hat{U}_j(\zeta) + B\hat{U}_k(\zeta) + \hat{\epsilon}(u,\zeta), \tag{5.26}$$

where we may suppose the branch of $f^{\frac{1}{4}}(u,z)$ to be the same as on the left hand side of (5.7). We shall also require the identity

$$\frac{\mathrm{d}\zeta}{\mathrm{d}z} = (-)^j \frac{2}{m} \frac{f^{\frac{1}{2}}(u,z)}{\zeta^{\frac{1}{2}(m-2)}}, \quad (\zeta \in \boldsymbol{S}_j), \tag{5.27}$$

in which the branch of $f^{\frac{1}{2}}(u,z)$ is the same as that used in (5.2) in constructing $\xi_c(u,z)$, and $\zeta^{\frac{1}{2}(m-2)}$ denotes the square of the branch of $\zeta^{\frac{1}{4}(m-2)}$ in (5.25). Equation (5.27) is obtained by dividing the equations

$$\mathrm{d}\xi_c/\mathrm{d}z = f^{\frac{1}{2}}(u,z), \quad \mathrm{d}\xi_c/\mathrm{d}\zeta = (-)^j\, \tfrac{1}{2}m\zeta^{\frac{1}{2}(m-2)},$$

obtained from (3.3) and (5.2).

Let us again denote by $\alpha_j$ the value of $\zeta$ corresponding to $z = a_j$, and at first suppose that $\alpha_j$ is finite (and therefore, also, $\xi_c(u,a_j)$ is finite). If $z \to a_j$ then $\mathscr{V}_{a_j,z}(H) \to 0$, and from (3.13) it follows that $\hat{\epsilon}(u,\alpha_j) = 0$. Substituting in (5.26) by means of this result and referring to (5.7), we obtain

$$A\hat{U}_j(\alpha_j) + B\hat{U}_k(\alpha_j) = (\tfrac{1}{2}m)^{-\frac{1}{2}}(1+\eta)\,\mathrm{e}^{-u\xi_c(u,a_j)}.$$

Similarly, on differentiating (5.26) with the aid of (5.27) and comparing the result with (5.8) as $z \to a_j$, we find that

$$A\hat{U}'_j(\alpha_j) + B\hat{U}'_k(\alpha_j) = (-)^{j-1}(\tfrac{1}{2}m)^{\frac{1}{2}}(1+\eta^1)\, u\alpha_j^{\frac{1}{2}(m-2)}\,\mathrm{e}^{-u\xi_c(u,a_j)}.$$

Solving the last two equations for $A$ and $B$ with the aid of the Wronskian relation

$$\hat{U}_j(\alpha_j)\,\hat{U}'_k(\alpha_j) - \hat{U}_k(\alpha_j)\,\hat{U}'_j(\alpha_j) = m\mathrm{i}\,\mathrm{e}^{-(j+k)\pi\mathrm{i}/m}\lambda_{j,k}\,u^{2/m}\alpha_j^{\frac{1}{2}(m-2)}$$

obtained from (2.16) and (5.25), we arrive at

$$A = \frac{\mathrm{e}^{(j+k)\pi\mathrm{i}/m}}{m\mathrm{i}\lambda_{j,k}\,u^{2/m}}\,\mathrm{e}^{-u\xi_c(u,a_j)}\{(\tfrac{1}{2}m)^{-\frac{1}{2}}(1+\eta)\,\alpha_j^{\frac{1}{2}(2-m)}\hat{U}'_k(\alpha_j) + (-)^j(\tfrac{1}{2}m)^{\frac{1}{2}}(1+\eta^1)\,u\hat{U}_k(\alpha_j)\},$$

$$B = \frac{\mathrm{i}\mathrm{e}^{(j+k)\pi\mathrm{i}/m}}{m\lambda_{j,k}\,u^{2/m}}\,\mathrm{e}^{-u\xi_c(u,a_j)}\{(\tfrac{1}{2}m)^{-\frac{1}{2}}(1+\eta)\,\alpha_j^{\frac{1}{2}(2-m)}\hat{U}'_j(\alpha_j) + (-)^j(\tfrac{1}{2}m)^{\frac{1}{2}}(1+\eta^1)\,u\hat{U}_j(\alpha_j)\}.$$

We now refer to (5.25) and substitute for $\hat{U}_j(\alpha_j)$ and $\hat{U}'_j(\alpha_j)$ by means of (2.12) and (2.13), and for $\hat{U}_k(\alpha_j)$ and $\hat{U}'_k(\alpha_j)$ by means of (2.23), (2.24) and (2.25), with the symbols $j$ and $k$ interchanged in the case of the last three equations. This analysis yields

$$A = (\tfrac{1}{2}m)^{-\frac{1}{2}}\mathrm{i}^{-j}\mathrm{e}^{j\pi\mathrm{i}/m}u^{(m-2)/(2m)}(1+\hat{A}), \tag{5.28}$$

$$B = (\tfrac{1}{2}m)^{-\frac{1}{2}}\mathrm{i}^{-j-1}\mathrm{e}^{k\pi\mathrm{i}/m}\lambda_{j,k}^{-1}\,u^{(m-2)/(2m)}\hat{B}, \tag{5.29}$$

where $$\hat{A} = \tfrac{1}{2}(1+\eta)\,\{1+\vartheta^1_{j\pm1}(u^{2/m}\alpha_j)\} + \tfrac{1}{2}(1+\eta^1)\,\{1+\vartheta_{j\pm1}(u^{2/m}\alpha_j)\} - 1 \mp \mathrm{i}\lambda_{j,k}^{-1}\lambda_{j,k\mp1}\hat{B}, \tag{5.30}$$

$$\hat{B} = \mathrm{e}^{-2u\xi_c(u,a_j)}[\tfrac{1}{2}(1+\eta)\{1+\vartheta^1_j(u^{2/m}\alpha_j)\} - \tfrac{1}{2}(1+\eta^1)\,\{1+\vartheta_j(u^{2/m}\alpha_j)\}]. \tag{5.31}$$

In (5.30) the upper or lower signs are taken according as $a_j$ lies in the left or right part of $\boldsymbol{D}_j$.



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Alternatively, assume that $\alpha_j$ is infinite. Then $\xi_c(u, a_j) = \infty$ and from lemma 5.1 we know that we must have $\eta^1 = \eta$. First, if $\mathrm{Re}\,\xi_c(u, a_j) = +\infty$ then $B = 0$; compare remark (*b*) in §3.3. On forming the ratio of the two sides of (5.26) and letting $z \to a_j$ we find that (5.28) applies with $\hat{A} = \eta$. Secondly, if $\mathrm{Re}\,\xi_c(u, a_j)$ is finite then $\mathrm{Im}\,\xi_c(u, a_j)$ must be infinite. By using similar analysis we again conclude that $B = 0$ and $\hat{A} = \eta$. Since the terms $\vartheta_j(u^{2/m}\alpha_j)$, $\vartheta_j^1(u^{2/m}\alpha_j)$, $\vartheta_{j\pm1}(u^{2/m}\alpha_j)$ and $\vartheta_{j\pm1}^1(u^{2/m}\alpha_j)$ all vanish when $\alpha_j = \infty$, these results show that (5.30) and (5.31) may be replaced by their limiting forms as $\alpha_j \to \infty$.

Bounds for $\hat{A}$ and $\hat{B}$ may be constructed from (5.30) and (5.31) by use of the hypotheses $|\eta|, |\eta^1| \leqslant \hat{\delta}$ and the bound for $\vartheta_j(u^{2/m}\alpha_j)$ and the other error terms obtained from (2.14). In this way we derive

$$|(1+\eta)\{1+\vartheta_{j\pm1}^1(u^{2/m}\alpha_j)\}-1|, \quad |(1+\eta^1)\{1+\vartheta_{j\pm1}(u^{2/m}\alpha_j)\}-1| \leqslant (1+\hat{\delta})\{1+\Theta(u\alpha_j^{\frac{1}{2}m})\}-1,$$

and $$|(1+\eta)\{1+\vartheta_j^1(u^{2/m}\alpha_j)\}-(1+\eta^1)\{1+\vartheta_j(u^{2/m}\alpha_j)\}| \leqslant 2(1+\hat{\delta})\{1+\Theta(u\alpha_j^{\frac{1}{2}m})\}-2.$$

Hence, whether or not $\xi_c(u, a_j)$ be finite, we have

$$|\hat{A}| \leqslant X_{j,k}(u), \quad |\hat{B}| \leqslant T_j \Delta_j(u)\,|\mathrm{e}^{-2u\xi_c(u, a_j)}|, \tag{5.32}$$

where $\Delta_j(u)$, $X_{j,k}(u)$ and $T_j$ are defined in §5.2.

### 5.5. *Proof of theorem* 5.1: *conclusion*

Theorem 3.1 shows that the representation (5.26) persists as $z$ passes along $\mathscr{P}$ and approaches $a_k$. To evaluate the error bounds (3.13) we note that when $v \in \mathscr{Q}$ we have, from (2.29),

$$\Omega(u^{2/m}v)\,E_{j,k}^{-\frac{1}{2}}(u^{2/m}v)\,M_{j,k}(u^{2/m}v)\,|U_j(u^{2/m}v)| \leqslant \Omega(u^{2/m}v)\,M_{j,k}^{\frac{2}{2}}(u^{2/m}v) \leqslant \rho_{j,k};$$

compare (3.9). If $|\xi_c(u, a_j)| = \infty$ then, as shown above, $B = 0$. Alternatively, if $\xi_c(u, a_j)$ is finite then we have

$$\Omega(u^{2/m}v)\,E_{j,k}^{-\frac{1}{2}}(u^{2/m}v)\,M_{j,k}(u^{2/m}v)\,|\,U_k(u^{2/m}v)\,\mathrm{e}^{-2u\xi_c(u, a_j)}|$$
$$\leqslant \Omega(u^{2/m}v)\,M_{j,k}^{\frac{2}{2}}(u^{2/m}v)\,E_{j,k}^{-\frac{2}{2}}(u^{2/m}v)\,E_{j,k}^{\frac{2}{2}}(u^{2/m}\alpha_j).$$

The right hand side is bounded by $\rho_{j,k}$, because $E_{j,k}(u^{2/m}v)$ is non-decreasing on $\mathscr{Q}$. Referring to (3.14), (5.28) and (5.29) we deduce that

$$\sigma_{j,k}/\rho_{j,k} \leqslant (\tfrac{1}{2}m)^{-\frac{1}{2}}u^{(m-2)/(2m)}(|1+\hat{A}|+|\lambda_{j,k}^{-1}\,\mathrm{e}^{2u\xi_c(u, a_j)}\hat{B}|),$$

and hence, with the aid of (5.32),

$$\sigma_{j,k}/\rho_{j,k} \leqslant (\tfrac{1}{2}m)^{-\frac{1}{2}}u^{(m-2)/(2m)}\{1+X_{j,k}(u)+T_j|\lambda_{j,k}^{-1}|\,\Delta_j(u)\},$$

this inequality holding whether or not $\xi_c(u, a_j)$ be finite. On substituting in (3.13) by means of this result and referring to (2.37), (2.43), (5.6) and (5.25), we perceive that

$$(\tfrac{1}{2}m)^{\frac{1}{2}}|\mathrm{e}^{-u\xi_c(u,z)}\hat{\epsilon}(u,\zeta)|, \quad (\tfrac{1}{2}m)^{-\frac{1}{2}}u^{-1}|\,\mathrm{e}^{-u\xi_c(u,z)}\zeta^{\frac{1}{2}(2-m)}\hat{\epsilon}'(u,\zeta)| \leqslant \Lambda_{j,k}(u,z)\,[1+\Theta\{u\xi_c(u,z)\}], \tag{5.33}$$

valid when $z \in \mathscr{P} \cap \mathbf{D}_k$, where we have denoted $\partial\hat{\epsilon}(u,\zeta)/\partial\zeta$ by $\hat{\epsilon}'(u,\zeta)$, for brevity.

The remaining step is to identify the representation (5.26), and its $z$-derivative, in the explicit forms given by the theorem in cases (i), (ii) and (iii). In (5.26) we substitute for $A$ and $B$ by means of (5.28) and (5.29), and for $\hat{U}_j(\zeta)$ and $\hat{U}_k(\zeta)$ by means of (5.25), (2.12), (2.23) and (2.24), with the symbol $j$ replaced by $k$ in the case of (2.12). This yields

$$f^{\frac{1}{4}}(u,z)\,w(u,z) = \mathrm{i}^{k-j-1}(1+\hat{A})\,[\lambda_{j,k}\,\mathrm{e}^{u\xi_c(u,z)}\{1+\vartheta_{k\pm1}(u^{2/m}\zeta)\} \pm \mathrm{i}\lambda_{j,k\pm1}\,\mathrm{e}^{-u\xi_c(u,z)}\{1+\vartheta_k(u^{2/m}\zeta)\}]$$
$$+\mathrm{i}^{k-j-1}\hat{B}\lambda_{j,k}^{-1}\,\mathrm{e}^{-u\xi_c(u,z)}\{1+\vartheta_k(u^{2/m}\zeta)\}+(\tfrac{1}{2}m)^{\frac{1}{2}}\hat{\epsilon}(u,\zeta), \tag{5.34}$$

the upper or lower signs being taken according as $z$ lies in the left or right part of $\boldsymbol{D}_k$. Next, if we differentiate (5.26) by use of (5.27), with the symbol $j$ replaced by $k$, and use similar analysis we arrive at

$$\begin{aligned} u^{-1}f^{-\frac{1}{4}}(u,z)\,\partial\{f^{\frac{1}{4}}(u,z)\,w(u,z)\}/\partial z \\ = \mathrm{i}^{k-j-1}(1+\hat{A})\,[\lambda_{j,k}\,\mathrm{e}^{u\xi_c(u,z)}\{1+\vartheta^1_{k\pm1}(u^{2/m}\zeta)\}\mp\mathrm{i}\lambda_{j,k\pm1}\,\mathrm{e}^{-u\xi_c(u,z)}\{1+\vartheta^1_k(u^{2/m}\zeta)\}] \\ -\mathrm{i}^{k-j-1}\hat{B}\lambda^{-1}_{j,k}\,\mathrm{e}^{-u\xi_c(u,z)}\{1+\vartheta^1_k(u^{2/m}\zeta)\}+(-)^k u^{-1}(\tfrac{1}{2}m)^{-\frac{1}{2}}\zeta^{\frac{1}{2}(2-m)}\hat{\epsilon}'(u,\zeta), \end{aligned} \tag{5.35}$$

with the same sign convention.

*Case* (i). We compare (5.9) and (5.34). Since there are two error terms in (5.9) there is a degree of freedom in the identification. The actual choice does not appear to be critical from the standpoint of applications. We shall adopt

$$\epsilon_{\pm1}(u,z) = (\tfrac{1}{2}m)^{\frac{1}{2}}\mathrm{i}^{-k+j+1}\mathrm{e}^{-u\xi_c(u,z)}\hat{\epsilon}(u,\zeta)+(1+\hat{A})\,\lambda_{j,k}\{1+\vartheta_{k\pm1}(u^{2/m}\zeta)\}-\lambda_{j,k}, \tag{5.36}$$

$$\epsilon_{\pm2}(u,z) = \{(1+\hat{A})\,\lambda_{j,k\pm1}\mp\mathrm{i}\hat{B}\lambda^{-1}_{j,k}\}\{1+\vartheta_k(u^{2/m}\zeta)\}-\lambda_{j,k\pm1}. \tag{5.37}$$

Similarly on comparing (5.10) and (5.35) we see that we can set

$$\epsilon^1_{\pm1}(u,z) = (\tfrac{1}{2}m)^{-\frac{1}{2}}\mathrm{i}^{k+j+1}u^{-1}\mathrm{e}^{-u\xi_c(u,z)}\zeta^{\frac{1}{2}(2-m)}\hat{\epsilon}'(u,\zeta)+(1+\hat{A})\,\lambda_{j,k}\{1+\vartheta^1_{k\pm1}(u^{2/m}\zeta)\}-\lambda_{j,k}, \tag{5.38}$$

$$\epsilon^1_{\pm2}(u,z) = \{(1+\hat{A})\,\lambda_{j,k\pm1}\mp\mathrm{i}\hat{B}\lambda^{-1}_{j,k}\}\{1+\vartheta^1_k(u^{2/m}\zeta)\}-\lambda_{j,k\pm1}. \tag{5.39}$$

*Case* (ii). We divide (5.13) and (5.34) by $\mathrm{e}^{u\xi_c(u,z)}$, let $z\to a_k$ and compare the results. Since $\mathrm{e}^{-2u\xi_c(u,z)}$, $\vartheta_k(u^{2/m}\zeta)$ and $\vartheta_{k\pm1}(u^{2/m}\zeta)$ all vanish in these circumstances, we obtain

$$\kappa_0 = (\tfrac{1}{2}m)^{\frac{1}{2}}\mathrm{i}^{-k+j+1}\lim_{z\to a_k}\{\mathrm{e}^{-u\xi_c(u,z)}\hat{\epsilon}(u,\zeta)\}+\hat{A}\lambda_{j,k}, \tag{5.40}$$

the existence of the limit on the right hand side being a consequence of theorem 12.1 of Olver (1974, ch. 6). Another consequence of the same theorem is that (5.14) is implied by (5.13).

*Case* (iii). The analysis in this case is more complicated, because although it is evident from (5.33) that $\mathrm{e}^{-u\xi_c(u,z)}\hat{\epsilon}(u,\zeta)$ is bounded in absolute value, this quantity does not tend to a limit as $z\to a_k$ along $\mathscr{P}$. Bearing in mind that we have $\mathrm{Re}\,\xi_c(u,z) = 0$ when $z\in\mathscr{P}\cap\boldsymbol{D}_k$, we proceed as follows.

By applying lemma 5.1 with $j = k$ and $k\pm1$ in turn, we see that when $z\to a_k$ along $\mathscr{P}$ the solution $w(u,z)$ can be expressed in the forms (5.16) and (5.17) in which the terms $\kappa_{\pm1}$ and $\kappa_{\pm2}$ are independent of $z$. On comparing (5.16) with the limiting form of (5.34), we obtain

$$\begin{aligned} \kappa_{\pm1}\,\mathrm{e}^{u\xi_c(u,z)}\pm\mathrm{i}\kappa_{\pm2}\,\mathrm{e}^{-u\xi_c(u,z)}+o(1) = \hat{A}\{\lambda_{j,k}\,\mathrm{e}^{u\xi_c(u,z)}\pm\mathrm{i}\lambda_{j,k\pm1}\,\mathrm{e}^{-u\xi_c(u,z)}\} \\ +\hat{B}\lambda^{-1}_{j,k}\,\mathrm{e}^{-u\xi_c(u,z)}+(\tfrac{1}{2}m)^{\frac{1}{2}}\mathrm{i}^{-k+j+1}\hat{\epsilon}(u,\zeta). \end{aligned}$$

Let $\{\zeta_n\}$, $n = 1, 2, 3, \ldots$ denote the infinite sequence of values of $\zeta$ on $\mathscr{Q}\cap\boldsymbol{S}_k$ for which the corresponding values of $\xi_c(u,z)$ are given by

$$2u\xi_c(u,z) = \pm n\pi\mathrm{i},$$

the upper or lower sign being taken according as $\mathscr{Q}$ coincides with the left or right boundary of $\boldsymbol{S}_k$. On letting $\zeta\to\alpha_k$ through the sequence $\{\zeta_{2n}\}$, we derive

$$\kappa_{\pm1}\pm\mathrm{i}\kappa_{\pm2} = \hat{A}(\lambda_{j,k}\pm\mathrm{i}\lambda_{j,k\pm1})+\hat{B}\lambda^{-1}_{j,k}+(-)^n(\tfrac{1}{2}m)^{\frac{1}{2}}\mathrm{i}^{-k+j+1}\hat{\epsilon}(u,\zeta_{2n})+o(1).$$

Similarly on letting $\zeta \to \alpha_k$ through the sequence $\{\zeta_{2n-1}\}$, we obtain

$$\kappa_{\pm 1} \mp i\kappa_{\pm 2} = \hat{A}(\lambda_{j,k} \mp i\lambda_{j,k\pm 1}) - \hat{B}\lambda_{j,k}^{-1} \pm i(-)^n (\tfrac{1}{2}m)^{\frac{1}{2}} i^{-k+j+1} \hat{\epsilon}(u, \zeta_{2n-1}) + o(1).$$

The last two equations may be solved for $\kappa_{\pm 1}$ and $\kappa_{\pm 2}$. In this way we find that

$$\kappa_{\pm 1} = (-)^n \tfrac{1}{2}(\tfrac{1}{2}m)^{\frac{1}{2}} i^{-k+j+1} \{\hat{\epsilon}(u, \zeta_{2n}) \pm i\hat{\epsilon}(u, \zeta_{2n-1})\} + \hat{A}\lambda_{j,k} + o(1), \tag{5.41}$$

$$\kappa_{\pm 2} = (-)^n \tfrac{1}{2}(\tfrac{1}{2}m)^{\frac{1}{2}} i^{-k+j+1} \{\mp i\hat{\epsilon}(u, \zeta_{2n}) - \hat{\epsilon}(u, \zeta_{2n-1})\} + \hat{A}\lambda_{j,k\pm 1} \mp i\hat{B}\lambda_{j,k}^{-1} + o(1). \tag{5.42}$$

Equations (5.36) to (5.42) supply exact formulae for the wanted error terms in the approximations (5.9), (5.10), (5.13), (5.14), (5.16) and (5.17). The desired inequalities (5.11), (5.12), (5.15), (5.18) and (5.19) immediately follow on substituting for the absolute values of $\vartheta_k(u^{2/m}\zeta)$, $\vartheta_k^1(u^{2/m}\zeta)$, $\vartheta_{k\pm 1}(u^{2/m}\zeta)$ and $\vartheta_{k\pm 1}^1(u^{2/m}\zeta)$ by means of their common bound $\Theta\{u\xi_c(u, z)\}$ obtained from (2.14), for $\hat{A}$ and $\hat{B}$ by means of (5.32), and for the terms involving values, or limiting values, of $\hat{\epsilon}(u, \zeta)$ and $\hat{\epsilon}'(u, \zeta)$ by means of (5.33). This completes the proof of theorem 5.1.

### 5.6. *A special case*

An important case of theorem 5.1 arises when both $\xi_c(u, a_j)$ and $\xi_c(u, a_k)$ are at infinity. Since several simplifications then become available, we state the result as a separate theorem. This may be compared with corresponding results for real variables given by Olver (1977*a*, §4).

THEOREM 5.2. *Assume the conditions of theorem* 5.1 *and also that* $|\xi_c(u, z)| \to \infty$ *as* $z \to a_j$ *or* $a_k$ *along* $\mathscr{P}$. *Then the solution of* (3.1) *defined by*

$$w(u, z) \sim f^{-\frac{1}{4}}(u, z)\, e^{-u\xi_c(u, z)} \quad (z \to a_j \text{ along } \mathscr{P}), \tag{5.43}$$

*is unique. Furthermore if* $z \to a_k$ *along* $\mathscr{P}$, *then* $w(u, z)$ *has the properties* (5.13) *and* (5.14) *when* $\operatorname{Re} \xi_c(u, a_k) = \infty$, *or the properties* (5.16) *and* (5.17) *when* $\operatorname{Re} \xi_c(u, a_k) = 0$, *where*

$$|\kappa_0|, \quad |\kappa_{\pm 1}|, \quad |\kappa_{\pm 2}| \leqslant C_{j,k} \left[ \exp\left\{ \frac{\rho_{j,k}}{m|\lambda_{j,k}| u^{2/m}} \mathscr{V}_{\mathscr{P}}(H) \right\} - 1 \right], \tag{5.44}$$

*provided that, when* (5.13) *and* (5.14) *hold, the right hand side of* (5.44) *is less than* $|\lambda_{j,k}|$.

It may also be noted that by constructing a direct proof for this special case it is not difficult to sharpen the bound (5.44) for $|\kappa_0|$ (but not for $|\kappa_{\pm 1}|$ or $|\kappa_{\pm 2}|$) by replacing $C_{j,k}$ by $(1 + \lambda_{j,k}^2)^{\frac{1}{2}}$; compare (2.38).

## 6. Asymptotic forms of the connection theorem

### 6.1. *Assumptions*

By supposing that the functions $f$ and $g$ are independent of the parameter $u$, we may construct asymptotic estimates of the error terms in theorem 5.1 for large values of $u$ by application of the results of §4. The results we shall obtain in the present section are analogous to those for real variables given in Olver (1977*b*, §2) but the method of proof is somewhat different.

We consider the equation

$$d^2w/dz^2 = \{u^2 f(z) + g(z)\} w, \tag{6.1}$$

in which $z \in \boldsymbol{D}$, again a bounded or unbounded open complex domain occupying one or more Riemann sheets, and the functions $f(z)$ and $g(z)$ are independent of the positive parameter $u$. We suppose that $(z-c)^{2-m}f(z)$ is holomorphic and non-vanishing in $\boldsymbol{D}$, where $c$ is an interior point of $\boldsymbol{D}$ and $m$ is a positive constant. Also, $g(z)$ is holomorphic in $\boldsymbol{D}$, punctured at $c$, and

$$g(z) = O\{(z-c)^{\gamma-1}\} \quad (z \to c), \tag{6.2}$$

where $\gamma$ is a positive constant.

As in §5.1 we define

$$\xi_c(z) = \int_c^z f^{\frac{1}{2}}(t)\, \mathrm{d}t, \tag{6.3}$$

taking the branch that has non-negative real part. We also adopt the other conditions and definitions associated with theorem 5.1, with the specialization that the functions $f(z)$ and $g(z)$ are now independent of $u$; for example,

$$F(z) = \int \left\{ \frac{1}{f^{\frac{1}{4}}(z)} \frac{\mathrm{d}^2}{\mathrm{d}z^2}\left(\frac{1}{f^{\frac{1}{4}}(z)}\right) - \frac{g(z)}{f^{\frac{1}{2}}(z)} \right\} \mathrm{d}z. \tag{6.4}$$

The following conventions were introduced in Olver (1977*b*), and will be followed in this and subsequent sections. First, the *symbol* $\underset{(2)}{=}$ *is used to signify that a given equation is valid, and also the corresponding equation obtained by formal differentiation with respect to z ignoring the differentiation of all O-terms.* Secondly, *whenever an O-term appears in an equation it is understood to be uniform with respect to all values of z associated with that equation.* Thirdly, *the symbol* $\chi(u)$ *denotes an arbitrary positive function of u having the properties*

$$\chi(u) \to 0, \quad 1/\chi(u) = O(u) \quad (u \to \infty). \tag{6.5}$$

Lastly, let $a$ and $b$ be any two points on a path $\mathscr{P}$. Then following Olver (1974, p. 121) we denote the part of $\mathscr{P}$ that lies between $a$ and $b$ by $(a, b)_{\mathscr{P}}$ or $[a, b]_{\mathscr{P}}$, according as $a$ and $b$ are both excluded or both included. Similarly for $(a, b]_{\mathscr{P}}$ and $[a, b)_{\mathscr{P}}$.

## 6.2. *Connection theorem for paths*

THEOREM 6.1. (i) *Assume the conditions and notation of* §6.1 *and that j and k are integers satisfying condition* (i) *of* §3.3. *Let* $\mathscr{P}$ *be a progressive path lying in* $\boldsymbol{D}_j \cup \boldsymbol{D}_k$ *and* $a_j, b_j, b_k$ *and* $a_k$ *be points, in that order, on* $\mathscr{P}$, *none of which depends on u or coincides with c; furthermore,* $a_j$ *is in the closure of* $\boldsymbol{D}_j$, $a_k$ *is in the closure of* $\boldsymbol{D}_k$, $b_j \in \boldsymbol{D}_j$ *and* $b_k \in \boldsymbol{D}_k$.† *Assume also that both* $1/\xi_c(z)$ *and* $F(z)$ *are of bounded variation on* $(a_j, b_j]_{\mathscr{P}}$ *and* $[b_k, a_k)_{\mathscr{P}}$, *and that* $w(u,z)$ *is a solution of the differential equation* (6.1) *having the properties*

$$f^{\frac{1}{4}}(z)\, w(u,z) \underset{(2)}{=} \{1+O(\chi)\}\, \mathrm{e}^{-u\xi_c(z)} \quad (z \in (a_j, b_j]_{\mathscr{P}}), \tag{6.6}$$

*as* $u \to \infty$, *and*

$$\mathrm{e}^{u\xi_c(z)} f^{\frac{1}{4}}(z)\, w(u,z), \quad f^{-\frac{1}{2}}(z)\, \mathrm{e}^{u\xi_c(z)} \partial\{f^{\frac{1}{4}}(z)\, w(u,z)\}/\partial z \to \text{non-zero finite limits,} \tag{6.7}$$

*as* $z \to a_j$ *along* $\mathscr{P}$.

*Then on* $[b_k, a_k)_{\mathscr{P}}$ *the analytic continuation of* $w(u,z)$ – *obtained by passing around the neighbourhood of c from* $\boldsymbol{D}_j$ *to* $\boldsymbol{D}_k$ *in the same sense as the sign of* $(k-j)$ – *is given by*

$$w(u,z) = w_{\mathrm{I}}(u,z), \tag{6.8}$$

$$w(u,z) = w_{\mathrm{I},1}(u,z) + w_{\mathrm{L}}(u,z), \tag{6.9}$$

*or*

$$w(u,z) = w_{\mathrm{I},-1}(u,z) + w_{\mathrm{R}}(u,z), \tag{6.10}$$

† Thus $a_j$ and $a_k$ (but not $b_j$ and $b_k$) may be boundary points of $\boldsymbol{D}$, including the point at infinity.

*according as $b_k$ is interior to $\boldsymbol{D}_k$, $[b_k, a_k)_{\mathscr{P}}$ coincides with the left boundary of $\boldsymbol{D}_k$, or $[b_k, a_k)_{\mathscr{P}}$ coincides with the right boundary of $\boldsymbol{D}_k$. Here $w_{\mathrm{I}}(u,z)$, $w_{\mathrm{I},\pm 1}(u,z)$, $w_{\mathrm{L}}(u,z)$ and $w_{\mathrm{R}}(u,z)$ are solutions of* (6.1) *having the following asymptotic forms on $[b_k, a_k)_{\mathscr{P}}$ as $u \to \infty$:*

$$f^{\frac{1}{4}}(z)\, w_{\mathrm{I}}(u,z), \quad f^{\frac{1}{4}}(z)\, w_{\mathrm{I},1}(u,z), \quad f^{\frac{1}{4}}(z)\, w_{\mathrm{I},-1}(u,z) \underset{(2)}{=} \mathrm{i}^{k-j-1}\{\lambda_{j,k} + O(\hat{\chi}_m)\}\, \mathrm{e}^{u\xi_c(z)}, \tag{6.11}$$

$$f^{\frac{1}{4}}(z)\, w_{\mathrm{L}}(u,z) \underset{(2)}{=} \mathrm{i}^{k-j}\{\lambda_{j,k+1} + O(\hat{\chi}_m)\}\, \mathrm{e}^{-u\xi_c(z)}, \tag{6.12}$$

$$f^{\frac{1}{4}}(z)\, w_{\mathrm{R}}(u,z) \underset{(2)}{=} -\mathrm{i}^{k-j}\{\lambda_{j,k-1} + O(\hat{\chi}_m)\}\, \mathrm{e}^{-u\xi_c(z)}. \tag{6.13}$$

*In these relations $\lambda_{j,k}$ is defined by* (2.17) *and* (2.18), *$f^{\frac{1}{4}}(z)$ denotes the branch obtained from that used in* (6.6) *by analytic continuation in the same manner as for $w(u,z)$, and $\hat{\chi}_m \equiv \hat{\chi}_m(u)$ is defined by*

$$\hat{\chi}_m(u) = \max\{\chi(u), \psi_m(u)\}, \tag{6.14}$$

*where $\psi_m(u)$ is given by* (4.5) *and* (4.6).

(ii) *If, in addition, $g(z)$ is analytic at $c$, then in* (6.11), (6.12) *and* (6.13) *we may take*

$$\hat{\chi}_m(u) = \max\{\chi(u), \chi_m(u)\}, \tag{6.15}$$

*where $\chi_m(u)$ is given by* (4.8).

The proof of this theorem is given in the next subsection.

*Remarks.* (*a*) Although $w_{\mathrm{I}}(u,z)$, $w_{\mathrm{I},1}(u,z)$ and $w_{\mathrm{I},-1}(u,z)$ have the same asymptotic form (6.11), they are distinct solutions of (6.1).

(*b*) The given conditions permit $b_j$ to coincide with $a_j$, provided that $a_j$ is not on the boundary of $\boldsymbol{D}$. In this event $(a_j, b_j]_{\mathscr{P}}$ is to be interpreted simply as $a_j$. Similarly for $a_k$ and $b_k$.

(*c*) The left boundary of $\boldsymbol{D}_k$ is also the right boundary of $\boldsymbol{D}_{k+1}$. Consequently a simple consistency check is furnished on replacing $k$ by $k+1$ in (6.10) and comparing the result with (6.9) by means of (6.11), (6.12) and (6.13). Bearing in mind that $\xi_c(u,z)$ changes sign on crossing the boundary, we perceive that consistency is maintained. Similarly for the right boundary of $\boldsymbol{D}_k$.

When $g(u,z)$ is analytic at $z = c$, another consistency check is furnished by taking $m$ to be a positive integer, other than unity, and replacing $k$ by $k+m$. In effect, this describes a simple closed circuit of $c$ once in the positive sense, and therefore changes $f^{\frac{1}{4}}(u,z)$ by the factor $\mathrm{e}^{\frac{1}{2}(m-2)\pi\mathrm{i}}$. Using this result we easily verify that the net effect is to leave (6.8) to (6.13) unchanged, as we expect, since $w_j(u,z)$ is analytic at $c$ for these values of $m$.

(*d*) In the various cases that arise when $c$ is real, $f(z)$ is real on the real axis and $m$ is an integer exceeding unity, theorem 6.1 yields the same results as theorems 1, 2, 3 and 4 of Olver (1977*b*). This assertion is verifiable in a straightforward manner and details will not be included.

### 6.3. *Proof of theorem* 6.1

In consequence of the assumed conditions, equation (6.1) has a solution $\hat{w}(u,z)$ with the properties

$$f^{\frac{1}{4}}(z)\, \hat{w}(u,z) \underset{(2)}{=} \{1 + O(u^{-1})\}\, \mathrm{e}^{-u\xi_c(z)} \quad (z \in (a_j, b_j]_{\mathscr{P}}), \tag{6.16}$$

as $u \to \infty$, and

$$\mathrm{e}^{u\xi_c(z)} f^{\frac{1}{4}}(z)\, \hat{w}(u,z) \to 1, \quad f^{-\frac{1}{2}}(z)\, \mathrm{e}^{u\xi_c(z)} \partial\{f^{\frac{1}{4}}(z)\, \hat{w}(u,z)\}/\partial z \to -u, \tag{6.17}$$

as $z \to a_j$ along $\mathscr{P}$; compare §§11 and 12 of Olver (1974, ch. 6). If we define

$$w(u,z) = \{1 + O(\chi)\}\, \hat{w}(u,z), \tag{6.18}$$

where the term $O(\chi)$ is independent of $z$, then $w(u,z)$ is another solution of (6.1). On combining

(6.16) and (6.18) and observing that in consequence of (6.5) $O(u^{-1}) \subseteq O(\chi)$, we obtain (6.6) with the term $O(\chi)$ now depending on $z$. In other words, a solution $w(u,z)$ having the properties (6.6) and (6.7) certainly exists. This solution is obviously not unique, however, because of the freedom of choice of the term $O(\chi)$ in (6.18).

Next, it is easily verified that all the conditions of theorem 5.1 are satisfied. On comparing (5.7) and (5.8) with (6.6) and (6.7) we see that we may set $\hat{\delta} = O(\chi)$ as $u \to \infty$. In $[b_k, a_k)_{\mathscr{P}}$ the wanted solution $w(u,z)$ is given by (5.9) and (5.10), and we now seek uniform asymptotic estimates for the error terms in these representations when $u$ is large.

From (5.4), (5.5) and the facts that

$$\Theta\{u\xi_c(a_j)\} = O(u^{-1}), \quad |e^{-2u\xi_c(a_j)}| \leqslant 1,$$

we obtain $\Delta_j(u) = O(\chi)$ and $X_{j,k}(u) = O(\chi)$. If we assume the conditions of part (i) of theorem 6.1, then theorem 4.1 (i) is applicable; hence we have $u^{-2/m}\mathscr{V}_{\mathscr{P}}(H) = O(\psi_m)$. Since $\mathscr{V}_{a_j,z}(H) \leqslant \mathscr{V}_{\mathscr{P}}(H)$ when $z \in [b_k, a_k)_{\mathscr{P}}$, we deduce from (5.6) that $\Lambda_{j,k}(u,z) = O(\psi_m)$ uniformly with respect to $z \in [b_k, a_k)_{\mathscr{P}}$. We also have $\Theta\{u\xi_c(z)\} = O(u^{-1})$ in the same circumstances.

Substituting in the right hand sides of (5.11) and (5.12) by means of the estimates found in the preceding paragraph, we conclude that

$$\epsilon_{\pm 1}(u,z), \quad \epsilon^1_{\pm 1}(u,z), \quad \epsilon_{\pm 2}(u,z), \quad \epsilon^1_{\pm 2}(u,z) = O(\hat{\chi}_m) \quad (z \in [b_k, a_k)_{\mathscr{P}}), \tag{6.19}$$

where $\hat{\chi}_m$ is defined by (6.14). The required results (6.11), (6.12) and (6.13) are obtained by combining (5.9) and (5.10) with (6.19), and also, when $b_k$ is interior to $\mathbf{D}_k$, absorbing the terms

$$\pm\, i^{k-j}\{\lambda_{j,k\pm 1} + \epsilon_{\pm 2}(u,z)\}\, e^{-u\xi_c(z)}, \quad \mp\, i^{k-j}\{\lambda_{j,k\pm 1} + \epsilon^1_{\pm 2}(u,z)\}\, e^{-u\xi_c(z)}$$

in the uniform error estimate $O(\hat{\chi}_m)\, e^{u\xi_c(z)}$, since in this event $\operatorname{Re}\xi_c(z) \geqslant \operatorname{Re}\xi_c(b_k) > 0$.

To complete the proof of part (i) of the theorem we have to show that the individual components $w_{\mathrm{I},1}(u,z)$, $w_{\mathrm{L}}(u,z)$, $w_{\mathrm{I},-1}(u,z)$ and $w_{\mathrm{R}}(u,z)$ in (6.9) and (6.10) can be chosen in such a way that in addition to possessing the asymptotic forms exhibited by (6.11) to (6.13) each is an exact solution of (6.1). In the case of $w_{\mathrm{I},1}(u,z)$ and $w_{\mathrm{L}}(u,z)$ this is verifiable in the following way.

From L.G. theory we know that there exist independent solutions $w_+(u,z)$ and $w_-(u,z)$, say, of (6.1) with the properties

$$f^{\frac{1}{4}}(z)\, w_+(u,z) \underset{(2)}{=} \{1 + O(u^{-1})\}\, e^{u\xi_c(z)}, \quad f^{\frac{1}{4}}(z)\, w_-(u,z) \underset{(2)}{=} \{1 + O(u^{-1})\}\, e^{-u\xi_c(z)}, \tag{6.20}$$

when $z \in [b_k, a_k)_{\mathscr{P}}$; compare (6.16). Let

$$f^{\frac{1}{4}}(z)\, w(u,z) = Af^{\frac{1}{4}}(z)\, w_+(u,z) + Bf^{\frac{1}{4}}(z)\, w_-(u,z), \tag{6.21}$$

where $A$ and $B$ are independent of $z$. To find these coefficients we substitute on the left hand side by means of (6.9), (6.11) and (6.12), and on the right hand side by means of (6.20). This gives one equation for $A$ and $B$. A second equation is found in a similar way from the differentiated form of (6.21). Solving the two equations, and recalling that $|e^{\pm u\xi_c(z)}| = 1$ in the present circumstances, we find that

$$A = i^{k-j-1}\{\lambda_{j,k} + O(\hat{\chi}_m)\}, \quad B = i^{k-j}\{\lambda_{j,k+1} + O(\hat{\chi}_m)\}. \tag{6.22}$$

On combining these results with (6.20) we see that $Af^{\frac{1}{4}}(z)\, w_+(u,z)$ and $Bf^{\frac{1}{4}}(z)\, w_-(u,z)$ have the properties (6.11) and (6.12) respectively, when $z \in [b_k, a_k)_{\mathscr{P}}$.

A similar analysis may be constructed for $w_{\mathrm{I},-1}(u,z)$ and $w_{\mathrm{R}}(u,z)$. This concludes the proof of part (i) of the theorem.



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To prove part (ii) we observe that if $g(z)$ is analytic at $c$, then theorem 4.1 (ii) permits $\psi_m$ to be replaced by $\chi_m$ everywhere; in particular (6.15) may be used instead of (6.14).

### 6.4. *Connection theorem for domains*

Theorem 6.1 supplies the L.G. form of the solution $w(u, z)$ when $z$ is confined to a path. In some applications we need asymptotic estimates for $w(u, z)$ that are uniform in subregions of $\boldsymbol{D}_k$. To construct a result of this nature we adopt the conditions of theorem 6.1, but now suppose that $|\xi_c(a_k)| = \infty$ and in $\boldsymbol{D}_k$ the path $\mathscr{P} \equiv \mathscr{P}_\theta$ coincides with the curve having the equation $\operatorname{ph} \xi_c(z) = \theta$, where $\theta$ is a parameter in the closed interval $[\theta_1, \theta_2]$. Here $\theta_1$ and $\theta_2$ are constants such that $-\frac{1}{2}\pi \leqslant \theta_1 < \theta_2 \leqslant \frac{1}{2}\pi$. In consequence, the point $b_k \equiv b_k(\theta)$ depends on $\theta$. We assume that $|\xi_c\{b_k(\theta)\}|$ is bounded and bounded away from zero, and also that

$$\mathscr{V}_{z, a_k}(F) \to 0 \quad (z \to a_k), \tag{6.23}$$

uniformly with respect to $\theta$, where the variation is evaluated along $\mathscr{P}_\theta$.

Theorem 6.2. *With the assumptions of this subsection, the solution $w(u, z)$ of* (6.1) *having the property* (6.6) *has the following asymptotic form on $[b_k(\theta), a_k)_{\mathscr{P}_\theta}$:*

$$f^{\frac{1}{4}}(z)\, w(u, z) \underset{(2)}{=} \mathrm{i}^{k-j-1}\{\lambda_{j,k} + O(\hat{\chi}_m)\}\, \mathrm{e}^{u\xi_c(z)} \pm \mathrm{i}^{k-j}\{\lambda_{j,k\pm 1} + O(\hat{\chi}_m)\}\, \mathrm{e}^{-u\xi_c(z)}, \tag{6.24}$$

*as $u \to \infty$, the upper or lower signs being taken throughout according as $\theta \geqslant 0$ or $\theta \leqslant 0$.*† *Furthermore, the O-terms are uniform with respect to $\theta$ (as well as $z$).*

This result is based on theorem 4.2, and the modifications to the proof of theorem 6.1 are quite straightforward.

*Remark.* If, in addition, we restrict $[b_k(\theta), a_k)_{\mathscr{P}_\theta}$ to an internal part of $\boldsymbol{D}_k$ (§5.1) then the contribution of the whole of the second term on the right hand side of (6.24) is absorbable in the uniform error estimate $O(\hat{\chi}_m)\, \mathrm{e}^{u\xi_c(z)}$ included in the first term, because $|\mathrm{e}^{-2u\xi_c(z)}|$ is exponentially and uniformly small in these circumstances.

## 7. Application of the connection theorem

### 7.1. *Classification of cases*

Assume $\boldsymbol{D}$ to be a simply connected open domain that contains $n$ transition points, and let the L.G. approximation of a solution of the differential equation (6.1) be given at a point in the closure of $\boldsymbol{D}$, other than one of the transition points. Then the L.G. approximation of the same solution at any other point, except a transition point, can be found by at most $n$ applications of theorem 6.1. Or, if we require the L.G. approximation of the solution in any appropriate subregion of $\boldsymbol{D}$, then it suffices to make $n-1$ or fewer applications of theorem 6.1, followed by one application of theorem 6.2. The process is completely illustrated by the case in which $\boldsymbol{D}$ contains just two transition points $c$ and $\hat{c}$, say, of multiplicities $m-2$ and $\hat{m}-2$, respectively, $m$ and $\hat{m}$ being arbitrary positive numbers.

As in §§5.1 and 6.1 we associate with $c$ the function

$$\xi_c(z) = \int_c^z f^{\frac{1}{2}}(t)\, \mathrm{d}t, \tag{7.1}$$

† Again, either sign may be adopted when $\theta = 0$.

where the branch of the integral has non-negative real part and is continuous except where $\operatorname{Re}\xi_c(z) = 0$. The last equation defines a set of curves emanating from $c$ which we shall call the *c-principal curves*. In a similar way, if

$$\xi_{\hat{c}}(z) = \int_{\hat{c}}^{z} f^{\frac{1}{2}}(t)\,\mathrm{d}t, \tag{7.2}$$

with $\operatorname{Re}\xi_{\hat{c}}(z) \geqslant 0$, then the curves on which $\xi_{\hat{c}}(z)$ is discontinuous, given by $\operatorname{Re}\xi_{\hat{c}}(z) = 0$, will be called the *ĉ-principal curves*.

In the plane of the variable

$$\xi(z) = \int f^{\frac{1}{2}}(z)\,\mathrm{d}z, \tag{7.3}$$

in which the integration constant is arbitrary and the branch is chosen in a continuous manner, the $c$-principal curves and the $\hat{c}$-principal curves are mapped as straight lines parallel to the imaginary axis. The theory of conformal mapping shows that the $c$-principal curves intersect each other on the same Riemann sheet only at $c$; similarly the $\hat{c}$-principal curves intersect only at $\hat{c}$. Nor may a $c$-principal curve intersect a $\hat{c}$-principal curve on the same Riemann sheet. It is possible, however, for $c$ and $\hat{c}$ to be linked by a common principal curve; this occurs when the join of $\xi(c)$ and $\xi(\hat{c})$ is parallel to the imaginary axis. The absence or presence of a common principal curve gives rise to two distinct cases, which we designate I and II. They are discussed in turn in the next two subsections. Further details will become clear from the examples treated in part C.

### 7.2. *Case I: no common principal curve*

Figure 6 indicates the disposition of the two sets of principal curves. We denote the region bounded by four principal curves, two from each set, either by $\boldsymbol{D}_0$ or $\hat{\boldsymbol{D}}_0$, depending whether it is being associated with $c$ or $\hat{c}$ in the connection process. Starting from $\boldsymbol{D}_0$ we enumerate the successive regions bounded by adjacent $c$-principal curves as $\boldsymbol{D}_1, \boldsymbol{D}_2, \ldots$, or $\boldsymbol{D}_{-1}, \boldsymbol{D}_{-2}, \ldots$, depending whether we are proceeding in the positive or negative sense around $c$. Similarly, starting from $\hat{\boldsymbol{D}}_0$ we denote the successive regions bounded by adjacent $\hat{c}$-principal curves by $\hat{\boldsymbol{D}}_1, \hat{\boldsymbol{D}}_2, \ldots$, or $\hat{\boldsymbol{D}}_{-1}, \hat{\boldsymbol{D}}_{-2}, \ldots$, depending whether we are proceeding in the positive or negative sense around $\hat{c}$. If $m$ is a positive integer exceeding unity, then there are $m$ distinct regions $\boldsymbol{D}_j$. In other cases the $\boldsymbol{D}_j$ may be finite or infinite in number, and occupy two or more Riemann sheets. Similarly for the $\hat{\boldsymbol{D}}_j$.

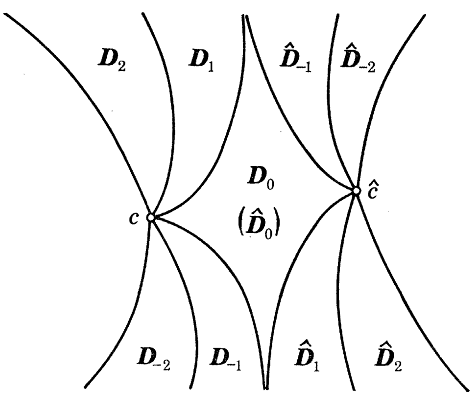


FIGURE 6. Case I: $z$-plane.

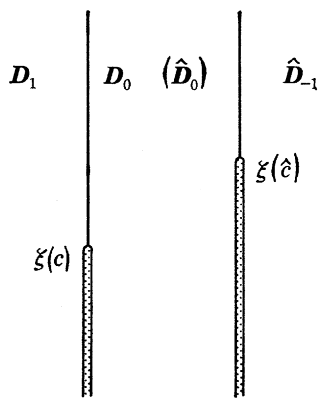


FIGURE 7. Case I: $\xi$-plane.

When $j \neq 0$ the regions $\boldsymbol{D}_j$ and $\hat{\boldsymbol{D}}_j$ are principal regions in the sense of §5.1. But $\boldsymbol{D}_0$ (or $\hat{\boldsymbol{D}}_0$) comprises only part of a principal region associated with $c$ (or $\hat{c}$). In the $\xi$-plane the maps of $\boldsymbol{D}_j$ and $\hat{\boldsymbol{D}}_j$ may extend over a half-plane when $j \neq 0$, but the map of $\boldsymbol{D}_0$ (or $\hat{\boldsymbol{D}}_0$) is confined to a vertical strip; compare figure 7.

Let us assume that we are given the L.G. approximation (6.6) of a solution $w(u,z)$ of (6.1), where $(a_j, b_j]_{\mathscr{P}}$ is a segment of a progressive path $\mathscr{P}$ in $\boldsymbol{D}_j$, $j$ being arbitrary. As special cases $(a_j, b_j]_{\mathscr{P}}$ may coincide with the left or right boundary of $\boldsymbol{D}_j$, or reduce to a single point in $\boldsymbol{D}_j$, other than a boundary point of $\boldsymbol{D}$. Our problem is to construct the L.G. approximation of the same solution on a segment $[\hat{b}_k, \hat{a}_k)_{\hat{\mathscr{P}}}$ of a progressive path $\hat{\mathscr{P}}$ in $\hat{\boldsymbol{D}}_k$, where $k$ is arbitrary. When $j = 0$ the problem is immediately solvable by regarding $\boldsymbol{D}_0$ as $\hat{\boldsymbol{D}}_0$, and passing from $\hat{\boldsymbol{D}}_0$ to $\hat{\boldsymbol{D}}_k$ by means of a single application of theorem 6.1, with $c$ replaced by $\hat{c}$. Similarly when $k = 0$.

When $j$ and $k$ are both non-zero we select an arbitrary point $a_0$ in the interior of $\boldsymbol{D}_0$ that can be joined to $b_j$ by an extension of $\mathscr{P}$ that passes through $c$ and is progressive, that is, $\operatorname{Re}\xi_c(z)$ is non-increasing as $z$ travels from $a_j$ to $c$ and non-decreasing as $z$ travels from $c$ to $a_0$. The L.G. form of $w(u,z)$ at $a_0$ is found by applying theorem 6.1 with $k = 0$, the relevant formulae being (6.8) and (6.11).

To prepare for passage through the second transition point we compare (7.1) and (7.2). Recalling that both $\operatorname{Re}\xi_c(z)$ and $\operatorname{Re}\xi_{\hat{c}}(z)$ are non-negative, we see that

$$\xi_c(z) + \xi_{\hat{c}}(z) = \xi_c(\hat{c}) = \xi_{\hat{c}}(c) \quad (z \in \boldsymbol{D}_0). \tag{7.4}$$

Hence $\operatorname{Re}\xi_c(z)$ is non-increasing on $[c, a_0]_{\mathscr{P}}$, *which implies that this segment of* $\mathscr{P}$ *is also a progressive path with respect to* $\hat{c}$. We now relabel $a_0$ as $\hat{a}_0$ and extend $\hat{\mathscr{P}}$, still progressive, to pass through $\hat{c}$ and continue to $\hat{a}_0$. The aggregate path $\mathscr{P} + \hat{\mathscr{P}}$ is indicated by the broken curve in figure 8. Using the

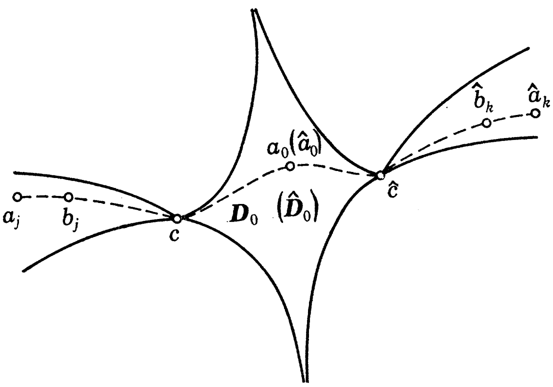


FIGURE 8. Case I: $z$-plane. --- progressive path.

L.G. form of $w(u,z)$ that is available at $\hat{a}_0$ after passage through $c$, we replace $e^{u\xi_c(z)}$ by $e^{u\xi_c(\hat{c})}\, e^{-u\xi_{\hat{c}}(z)}$; compare (7.4). Theorem 6.1 is then applied, with $c$ replaced by $\hat{c}$ and $j = 0$, and we select (6.8), (6.9) or (6.10), depending whether $b_k$ is an interior point of $\hat{\boldsymbol{D}}_k$, $[\hat{b}_k, \hat{a}_k)_{\hat{\mathscr{P}}}$ coincides with the left boundary of $\hat{\boldsymbol{D}}_k$ or $[\hat{b}_k, \hat{a}_k)_{\hat{\mathscr{P}}}$ coincides with the right boundary of $\hat{\boldsymbol{D}}_k$. The desired L.G. form on $[\hat{b}_k, \hat{a}_k)_{\hat{\mathscr{P}}}$ is then available from (6.11), (6.12) and (6.13).

### 7.3. *Case II: common principal curve*

Figure 9 indicates the disposition of the two sets of principal curves and the labelling of the corresponding regions. There are now two regions $\boldsymbol{D}_0$ ($\equiv \hat{\boldsymbol{D}}_1$) and $\boldsymbol{D}_1$ ($\equiv \hat{\boldsymbol{D}}_0$) that have both $c$ and

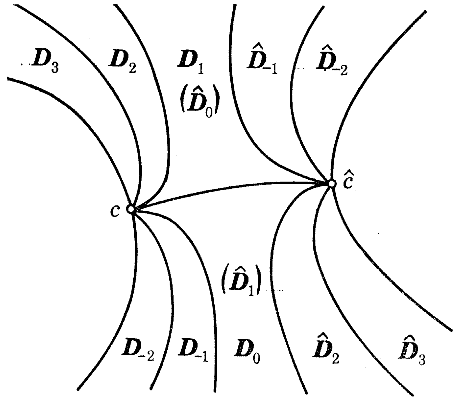


FIGURE 9. Case II: $z$-plane.

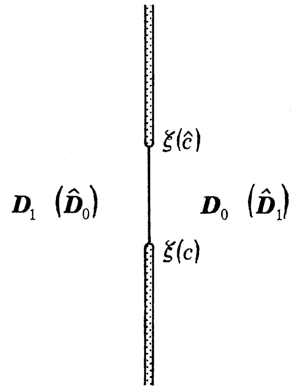


FIGURE 10. Case II: $\xi$-plane.

$\hat{c}$ on their boundaries; furthermore each is a principal region in the sense of § 5.1 with respect to either transition point. Part of the mapping on the $\xi$-plane is indicated in figure 10. Again, our problem is to connect the L.G. form (6.6) on a given segment $(a_j, b_j]_{\mathscr{P}}$ of a progressive path $\mathscr{P}$ in $\boldsymbol{D}_j$, with the L.G. form of the same solution on a segment $[\hat{b}_k, \hat{a}_k)_{\hat{\mathscr{P}}}$ of a progressive path $\hat{\mathscr{P}}$ in $\hat{\boldsymbol{D}}_k$, for arbitrary values of $j$ and $k$. As in case I, when $j$ or $k$ is zero or unity this problem can be solved by a single application of theorem 6.1. We therefore exclude these cases in the remainder of this subsection.

The essential difference from the previous case is that there is a change of sign in the relation between $\xi_c(z)$ and $\xi_{\hat{c}}(z)$: in place of (7.4) we now have

$$\xi_c(z) - \xi_{\hat{c}}(z) = \xi_c(\hat{c}) = -\xi_{\hat{c}}(c) \quad (z \in \boldsymbol{D}_0 \text{ or } \boldsymbol{D}_1). \tag{7.5}$$

In consequence, if we choose an arbitrary point $a_0$ in the interior of $\boldsymbol{D}_0$ and try to follow the method of I, we encounter a difficulty as soon as we prepare for passage through the second transition point. This is because $e^{u\xi_c(z)}$ is replaced by $e^{u\xi_c(\hat{c})}\,e^{u\xi_{\hat{c}}(z)}$, and the absence of a negative sign in the argument of the last exponential function precludes a second application of theorem 6.1.

To overcome the difficulty, we place $a_0$ on the common principal curve linking $c$ and $\hat{c}$, as indicated in figure 11. Then $[c, a_0]_{\mathscr{P}}$ coincides with this principal curve, and the L.G. form of

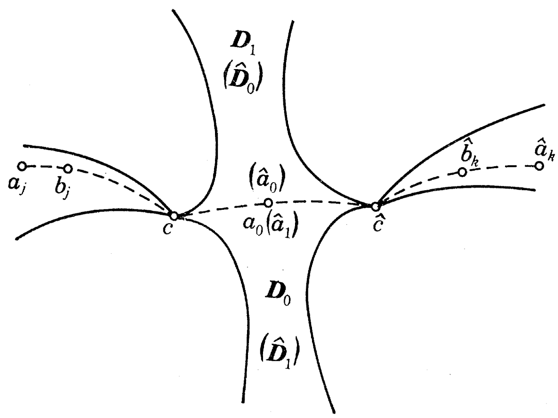


FIGURE 11. Case II: $z$-plane. --- progressive path.

$w(u,z)$ at $a_0$ is given by (6.9), (6.11) and (6.12) with $k = 0$. To prepare for passage through $\hat{c}$, we observe that $\boldsymbol{D}_0 \equiv \hat{\boldsymbol{D}}_1$. Consequently we relabel $a_0$ as $\hat{a}_1$, and extend $\hat{\mathscr{P}}$, still progressive, to pass through $\hat{c}$ and continue along the common principal curve until $\hat{a}_1$ is reached. The aggregate path $\mathscr{P} + \hat{\mathscr{P}}$ is indicated by the broken curve in figure 11. In the L.G. form of $w(u,z)$ at $\hat{a}_1$ found by passage through $c$, there are now two terms $w_{\mathrm{I},1}(u,z)$ and $w_{\mathrm{L}}(u,z)$. The contribution from $w_{\mathrm{L}}(u,z)$ to the L.G. form on $[\hat{b}_k, \hat{a}_k)_{\hat{\mathscr{P}}}$ is obtained by replacing $\mathrm{e}^{-u\xi_c(z)}$ by $\mathrm{e}^{-u\xi_c(\hat{c})}\,\mathrm{e}^{-u\xi_{\hat{c}}(z)}$ and applying theorem 6.1, with $c$ replaced by $\hat{c}$ and $j = 1$. To handle the contribution from $w_{\mathrm{I},1}(u,z)$, the key step is to regard $a_0$ as a member of $\hat{\boldsymbol{D}}_0$, and relabel it $\hat{a}_0$. Since this entails crossing a principal curve, $\xi_{\hat{c}}(z)$ is replaced by $-\xi_{\hat{c}}(z)$. Thus $\mathrm{e}^{u\xi_c(z)}$ becomes $\mathrm{e}^{u\xi_c(\hat{c})}\,\mathrm{e}^{-u\xi_{\hat{c}}(z)}$, where $\hat{c}$ is regarded as a member of $\boldsymbol{D}_0$ in calculating $\xi_c(\hat{c})$ (so that $\xi_c(\hat{c})$ is positive imaginary), and $z$ is regarded as a member of $\hat{\boldsymbol{D}}_0$. The contribution from $w_{\mathrm{I},1}(u,z)$ to the L.G. form on $[\hat{b}_k, \hat{a}_k)_{\hat{\mathscr{P}}}$ is then found by applying theorem 6.1, with $c$ replaced by $\hat{c}$ and $j = 0$.

### 7.4. *Remarks on cases I and II*

(*a*) Instead of using theorem 6.1 for passage through $\hat{c}$ in cases I and II, we may use theorem 6.2. This modification will yield a subregion (in place of curves) in $\hat{\boldsymbol{D}}_k$ in which the L.G. approximation is uniformly valid for large $u$.

(*b*) If, in case I, $k = \pm 1$ and $\hat{\mathscr{P}}$ lies in an internal part of $\hat{\boldsymbol{D}}_k$, then it may be possible to link $(a_j, b_j]_{\mathscr{P}}$ to $[\hat{b}_k, \hat{a}_k)_{\hat{\mathscr{P}}}$ by a path in $\boldsymbol{D}_j \cup \boldsymbol{D}_0 \cup \hat{\boldsymbol{D}}_k$ that does not intersect $\hat{c}$ and is progressive with respect to $c$ along its entire length. (For $k = -1$ this can be seen from the $\xi$-map depicted in figure 7). In this event a *single* application of theorem 6.1 suffices to trace the L.G. forms from $(a_j, b_j]_{\mathscr{P}}$ to $[\hat{b}_k, \hat{a}_k)_{\hat{\mathscr{P}}}$.

(*c*) When both transition points are of fractional multiplicity, it may happen that they are joined by a common principal curve on one Riemann sheet but not on others. In this event either the method of case I or that of case II may be employed (though the former is, of course, somewhat simpler).

# PART C. EXAMPLES

## 8. Example 1: four real turning points of differing multiplicities

### 8.1. *Topology of the principal regions*

Each of the six examples treated in §5 of Olver (1977*b*) can be solved by the methods of the present paper in all cases in which the coefficients in the differential equation are analytic functions of the independent variable. For the purpose of illustration we consider a special case of the fifth example. With a slight change of notation, this is given by

$$\mathrm{d}^2 w/\mathrm{d}z^2 = u^2 f(z)\, w, \tag{8.1}$$

where

$$f(z) = (z-c_1)^2 (z-c_2)\,(z-c_3)^4 (z-c_4)^3, \tag{8.2}$$

$c_1$, $c_2$, $c_3$ and $c_4$ being real constants such that $c_1 < c_2 < c_3 < c_4$. Thus there are turning points of multiplicities 2, 1, 4 and 3 at $c_1$, $c_2$, $c_3$ and $c_4$, respectively. The problem is to calculate the eigenvalues of the system for the interval $(-\infty, \infty)$, that is, the values of $u$ for which equation (8.1) admits a non-trivial solution that is recessive as $z \to \pm\infty$.

The numbers of principal curves that emerge from $c_1$, $c_2$, $c_3$ and $c_4$ are 4, 3, 6 and 5, respectively. Their configuration is indicated in figure 12, only one Riemann sheet being necessary. The points $c_2$ and $c_3$ are joined by a common principal curve along the real axis, as are $c_3$ and $c_4$. As $|z| \to \infty$



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the asymptotic form of the function $\xi(z)$ defined by (7.3) and (8.2) is given by $\xi(z) \sim \pm\frac{1}{6}z^6$. Hence the principal curves approach the point at infinity in the 12 directions specified by

$$\mathrm{ph}\, z = \tfrac{1}{12}(2s+1)\,\pi \quad (s = 0, 1, \ldots, 11).$$

Of the 14 curves that pass to infinity, one pair from $c_1$ and $c_2$ share the asymptotic direction $\frac{3}{4}\pi$ and another (conjugate) pair from the same points share the asymptotic direction $\frac{5}{4}\pi$.

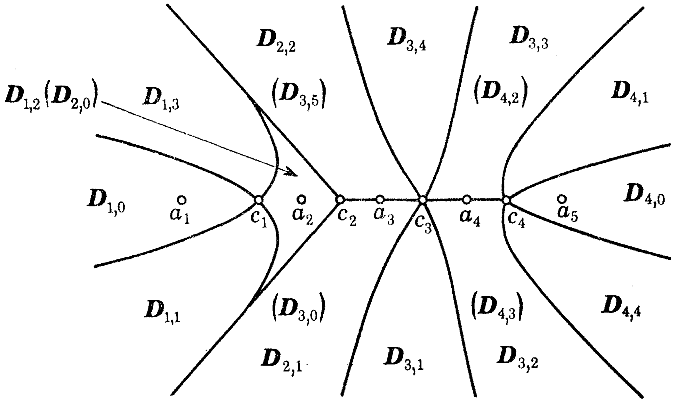


FIGURE 12. Example 1: $z$-plane.

The 16 principal curves divide the $z$-plane into 13 closed domains. Those with $c_1$ on the boundary are labelled $\boldsymbol{D}_{1,0}$, $\boldsymbol{D}_{1,1}$, $\boldsymbol{D}_{1,2}$ and $\boldsymbol{D}_{1,3}$ in the positive rotational sense; those with $c_2$ on the boundary are labelled $\boldsymbol{D}_{2,0}$, $\boldsymbol{D}_{2,1}$ and $\boldsymbol{D}_{2,2}$, and so on. These notations overlap; as indicated on the diagram $\boldsymbol{D}_{2,0} \equiv \boldsymbol{D}_{1,2}$, and so on. The $\xi$-map of each domain is a half-plane, except that of $\boldsymbol{D}_{2,0}$, which is a vertical strip.

### 8.2. *Construction of the connection formula*

Let $a_1$, $a_2$, $a_3$, $a_4$ and $a_5$ be arbitrarily chosen fixed points on the real axis such that

$$-\infty < a_1 < c_1 < a_2 < c_2 < a_3 < c_3 < a_4 < c_4 < a_5 < \infty.$$

By L.G. theory, all solutions of (8.1) that are recessive as $z \to -\infty$ are multiples of the solution $w(u, z)$ that is real on the real axis and has the properties

$$f^{\frac{1}{4}}(z)\, w(u, z) \underset{(2)}{=} \{1 + O(u^{-1})\}\, \mathrm{e}^{-u\xi_{c_1}(z)} \quad (z \in (-\infty, a_1])$$

as $u \to \infty$, and
$$f^{\frac{1}{4}}(z)\, w(u, z) \sim \mathrm{e}^{-u\xi_{c_1}(z)}$$

as $z \to -\infty$. The choice of branch of $f^{\frac{1}{4}}(z)$ in these relations is immaterial, as long as it is continuous. The L.G. form of this solution at $a_2$ is immediately found by applying theorem 6.1 with $c = c_1$, $\chi(u) = u^{-1}$, $j = 0$, $k = 2$ and $m = 4$. From (6.8) and (6.11), we obtain

$$f^{\frac{1}{4}}(z)\, w(u, z) \underset{(2)}{=} 2^{\frac{1}{2}}\mathrm{i}\{1 + O(\chi_4)\}\, \mathrm{e}^{u\xi_{c_1}(z)} \quad (z = a_2).$$

To prepare for passage through $c_2$, we follow the method of §7, case I and substitute in the last equation by means of the identity

$$\xi_{c_1}(z) = \xi_{c_1}(c_2) - \xi_{c_2}(z) \quad (z \in \boldsymbol{D}_{2,0}(\equiv \boldsymbol{D}_{1,2})),$$

to obtain
$$A_1(u)\, f^{\frac{1}{4}}(z)\, w(u, z) \underset{(2)}{=} \{1 + O(\chi_4)\}\, \mathrm{e}^{-u\xi_{c_2}(z)} \quad (z = a_2),$$

where
$$A_1(u) = -2^{-\frac{1}{2}}\mathrm{i}\, \mathrm{e}^{-u\xi_{c_1}(c_2)}.$$

At this stage we regard $a_3$ as belonging to the left boundary of $\boldsymbol{D}_{2,1}$, and therefore apply theorem 6.1 with $c = c_2$, $\chi = \chi_4$, $j = 0$, $k = 1$ and $m = 3$. Recalling that $O(\chi_3) \subset O(\chi_4)$, we obtain from (6.9), (6.11) and (6.12)

$$A_1(u) f^{\frac{1}{4}}(z)\, w(u,z) \underset{(2)}{=} \{1+O(\chi_4)\}\,\mathrm{e}^{u\xi_{c_2}(z)} + \mathrm{i}\{1+O(\chi_4)\}\,\mathrm{e}^{-u\xi_{c_2}(z)} \quad (z = a_3(\in \boldsymbol{D}_{2,1})).$$

To prepare for passage through $c_3$, we follow the method of § 7, case II and use the identity

$$\xi_{c_2}(z) = \xi_{c_2}(c_3) + \xi_{c_3}(z) \quad (z \in \boldsymbol{D}_{2,1});$$

compare (7.5). Substitution in the preceding relation produces

$$A_1(u) f^{\frac{1}{4}}(z)\, w(u,z) \underset{(2)}{=} A_2(u)\,\{1+O(\chi_4)\}\,\mathrm{e}^{u\xi_{c_3}(z)} + \mathrm{i}A_2^{-1}(u)\,\{1+O(\chi_4)\}\,\mathrm{e}^{-u\xi_{c_3}(z)} \quad (z = a_3(\in \boldsymbol{D}_{2,1}), \tag{8.3}$$

where $$A_2(u) = \mathrm{e}^{u\xi_{c_2}(c_3)}, \quad A_2^{-1}(u) = \mathrm{e}^{-u\xi_{c_2}(c_3)} \quad (c_3 \in \boldsymbol{D}_{2,1}). \tag{8.4}$$

Again, at this stage we regard $a_4$ as belonging to the left boundary of $\boldsymbol{D}_{3,2}$. The contribution from the second term on the right hand side of (8.3) is found by applying theorem 6.1 with $c = c_3$, $\chi = \chi_4$, $j = 0$, $k = 2$ and $m = 6$. From (6.9), (6.11) and (6.12) this contribution is calculated to be

$$\mathrm{i}A_2^{-1}(u)\,[3^{\frac{1}{2}}\mathrm{i}\{1+O(\chi_6)\}\,\mathrm{e}^{u\xi_{c_3}(z)} - 2\{1+O(\chi_6)\}\,\mathrm{e}^{-u\xi_{c_3}(z)}] \quad (z = a_4(\in \boldsymbol{D}_{3,2})). \tag{8.5}$$

To trace the contribution of the first term on the right hand side of (8.3), the method of § 7, case II requires us to regard $a_3$ as a left boundary point of $\boldsymbol{D}_{3,5}$. This replaces $\mathrm{e}^{u\xi_{c_3}(z)}$ by $\mathrm{e}^{-u\xi_{c_3}(z)}$. We then apply theorem 6.1 with $c = c_3$, $\chi = \chi_4$, $j = 5$, $k = 2$ and $m = 6$. From (6.9), (6.11) and (6.12) the contribution is found to be

$$A_2(u)\,[-2\{1+O(\chi_6)\}\,\mathrm{e}^{u\xi_{c_3}(z)} - 3^{\frac{1}{2}}\mathrm{i}\{1+O(\chi_6)\}\,\mathrm{e}^{-u\xi_{c_3}(z)}] \quad (z = a_4(\in \boldsymbol{D}_{3,2})). \tag{8.6}$$

Because $\xi_{c_2}(c_3)$ is purely imaginary it follows from (8.4) that $|A_2(u)| = 1$. Hence the result of combining (8.5) and (8.6) is expressible in the form

$$A_1(u) f^{\frac{1}{4}}(z)\, w(u,z) \underset{(2)}{=} \{A_3(u)+O(\chi_6)\}\,\mathrm{e}^{u\xi_{c_3}(z)} + \{A_4(u)+O(\chi_6)\}\,\mathrm{e}^{-u\xi_{c_3}(z)} \quad (z = a_4(\in \boldsymbol{D}_{3,2})),$$

where $$A_3(u) = -3^{\frac{1}{2}}A_2^{-1}(u) - 2A_2(u), \quad A_4(u) = -2\mathrm{i}A_2^{-1}(u) - 3^{\frac{1}{2}}\mathrm{i}A_2(u). \tag{8.7}$$

We pass through the remaining turning point $c_4$ in a similar manner to $c_3$. This calculation is slightly easier, because (6.8) is used in place of (6.9). The result is found to be

$$A_1(u) f^{\frac{1}{4}}(z)\, w(u,z) \underset{(2)}{=} -2\cos(\tfrac{1}{5}\pi)\,[\{A_4(u)+O(\chi_6)\}\,\mathrm{e}^{-u\xi_{c_3}(c_4)} + \mathrm{i}\{A_3(u)+O(\chi_6)\}\,\mathrm{e}^{u\xi_{c_3}(c_4)}]\,\mathrm{e}^{u\xi_{c_4}(z)} \quad (z \in [a_5, \infty)),$$

where $\xi_{c_3}(c_4)$ is calculated on the assumption that $c_4 \in \boldsymbol{D}_{3,2}$.

### 8.3. *Determination of the eigenvalues*

The condition that $w(u,z)$ be recessive as $z \to +\infty$ is that the content of the square brackets in the last equation is zero. (Compare the first footnote on page 689 of Olver 1977*b*.) Since $|\mathrm{e}^{\pm u\xi_{c_3}(c_4)}|$ is unity and $\chi_6 = u^{-\frac{2}{3}}$, we find that

$$A_3(u)\,\mathrm{e}^{u\xi_{c_3}(c_4)} - \mathrm{i}A_4(u)\,\mathrm{e}^{-u\xi_{c_3}(c_4)} = O(u^{-\frac{2}{3}}). \tag{8.8}$$

To express the last equation in real form, we have by definition $\xi_{c_2}(c_3) = \mathrm{i}\rho$, $\xi_{c_3}(c_4) = \mathrm{i}\sigma$, where

$$\rho = \int_{c_2}^{c_3} |f(t)|^{\frac{1}{2}}\,\mathrm{d}t, \quad \sigma = \int_{c_3}^{c_4} |f(t)|^{\frac{1}{2}}\,\mathrm{d}t.$$

On referring to (8.4) and (8.7), we find that (8.8) reduces to

$$3^{\frac{1}{2}}\cos\{(\sigma-\rho)u\}+2\cos\{(\sigma+\rho)u\} = O(u^{-\frac{2}{3}}),$$

or, equivalently,
$$\cos[\sigma u+\arctan\{\tan^2(\tfrac{1}{12}\pi)\tan(\rho u)\}] = O(u^{-\frac{2}{3}}). \tag{8.9}$$

Therefore
$$\sigma u+\arctan\{\tan^2(\tfrac{1}{12}\pi)\tan(\rho u)\} = (n+\tfrac{1}{2})\pi+O(u^{-\frac{2}{3}}), \tag{8.10}$$

where $n$ is an arbitrary integer. After making appropriate notational changes we see that the last equation is the same as (5.29) of Olver (1977*b*), and the rest of the analysis proceeds as in this reference.

## 9. Example 2: three triple turning points on the unit circle

### 9.1. *Topology of the principal regions*

In this section we consider the differential equation (8.1) with

$$f(z) = (z^3-1)^3. \tag{9.1}$$

The transition points are turning points of third order, situated at 1, $\omega$ and $1/\omega$, where $\omega = e^{\frac{2}{3}\pi i}$. Five principal curves emerge from each turning point, the phases of their initial directions being $\frac{1}{5}(2s+1)\pi$, $s$ being an integer. One of these curves is the real interval $(-\infty, 1]$. The configuration of the others is indicated in figure 13, only one Riemann sheet being necessary.

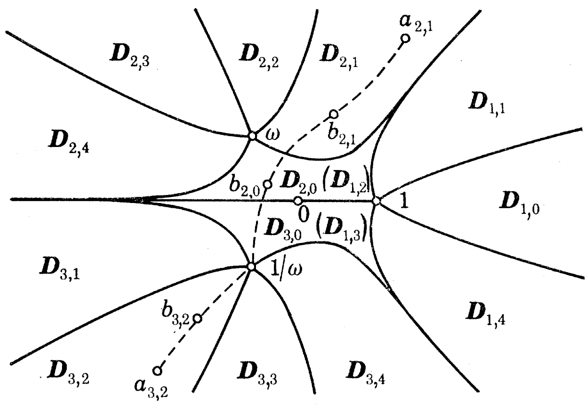


Figure 13. Example 2: $z$-plane. --- progressive path $\mathscr{P}$.

With $\xi(z)$ defined by (7.3) and (9.1), we have $\xi(z) \sim \pm\frac{2}{11}z^{\frac{11}{2}}$ as $|z|\to\infty$. Hence the principal curves approach the point at infinity in the 11 directions given by

$$\operatorname{ph} z = \tfrac{1}{11}(2s+1)\pi \quad (s = 0, 1, \ldots, 10).$$

Two curves, one from 1 and one from $\omega$, share the asymptotic direction $\frac{3}{11}\pi$; similarly two curves from 1 and $1/\omega$ share the asymptotic direction $\frac{19}{11}\pi$. Three curves, one from each turning point, approach infinity in the direction $\pi$.

The enumeration of the 13 closed domains bounded by the principal curves is done in a similar manner to that of example 1; the actual labelling we employ is indicated in figure 13. Each domain is mapped on the $\xi$-plane as a half-plane, except $\boldsymbol{D}_{1,2}$ $(\equiv \boldsymbol{D}_{2,0})$ and $\boldsymbol{D}_{1,3}$ $(\equiv \boldsymbol{D}_{3,0})$, which are mapped as vertical strips.

### 9.2. *Construction of the connection formulae*

Although there are three turning points in this example, at most two connections are needed to trace the L.G. forms of any given solution from one domain to another. For example, let us consider a solution that is recessive at infinity in $\boldsymbol{D}_{3,2}$. A single application of theorem 6.1 (or theorem 6.2) with $c = 1/\omega$ enables the L.G. form of this solution to be found in each of the following domains: $\boldsymbol{D}_{3,0}$, $\boldsymbol{D}_{3,1}$, $\boldsymbol{D}_{3,3}$, $\boldsymbol{D}_{3,4}$, the left part of $\boldsymbol{D}_{2,4}$, the right parts of $\boldsymbol{D}_{1,4}$ and $\boldsymbol{D}_{2,1}$, and a part of $\boldsymbol{D}_{2,0}$. To reach $\boldsymbol{D}_{1,0}$, $\boldsymbol{D}_{1,1}$ and the remaining parts of $\boldsymbol{D}_{1,4}$ and $\boldsymbol{D}_{2,0}$ two connections are needed, one via $1/\omega$ to an interior point of $\boldsymbol{D}_{3,0}$, then one via 1. Similarly two connections suffice to reach $\boldsymbol{D}_{2,2}$, $\boldsymbol{D}_{2,3}$ and the remaining parts of $\boldsymbol{D}_{2,1}$ and $\boldsymbol{D}_{2,4}$, one via $1/\omega$ to an interior point of $\boldsymbol{D}_{2,0} \cup \boldsymbol{D}_{3,0}$, then one via $\omega$.

For illustration, we consider in detail the problem of tracing the L.G. forms from $\boldsymbol{D}_{3,2}$ to $\boldsymbol{D}_{2,1}$. By considering the $\xi$-map it is easily seen that a doubly infinite path $\mathscr{P}$ can be found in the union of $\boldsymbol{D}_{3,2}$, $\boldsymbol{D}_{3,0}$, $\boldsymbol{D}_{2,0}$ and any internal part of $\boldsymbol{D}_{2,1}$ that is progressive with respect to the turning point at $1/\omega$ along its entire length. This path is indicated by the broken curve in figure 13. Let $a_{3,2}$ and $a_{2,1}$ be the points at infinity on $\mathscr{P}$ in $\boldsymbol{D}_{3,2}$ and $\boldsymbol{D}_{2,1}$, respectively, and $b_{3,2}$, $b_{2,0}$ and $b_{2,1}$ be points on $\mathscr{P}$ in the interiors of $\boldsymbol{D}_{3,2}$, $\boldsymbol{D}_{2,0}$ and $\boldsymbol{D}_{2,1}$, respectively. By L.G. theory, any solution of the differential equation that is recessive as $z \to \infty$ in $\boldsymbol{D}_{3,2}$ is a multiple of the solution $w(u,z)$ having the properties

$$\left.\begin{aligned} f^{\frac{1}{4}}(z)\, w(u,z) &\underset{(2)}{=} \{1+O(u^{-1})\}\, \mathrm{e}^{-u\xi_{1/\omega}(z)} \quad (z \in (a_{3,2}, b_{3,2}]_{\mathscr{P}}) \\ \text{as } u \to \infty, \text{ and} \qquad f^{\frac{1}{4}}(z)\, w(u,z) &\sim \mathrm{e}^{-u\xi_{1/\omega}(z)} \end{aligned}\right\} \tag{9.2}$$

as $z \to a_{3,2}$ along $\mathscr{P}$. Any continuous branch of $f^{\frac{1}{4}}(z)$ may be adopted in these relations. To find the asymptotic form of $w(u,z)$ on $[b_{2,1}, a_{2,1})_{\mathscr{P}}$, we apply theorem 6.1 with $c = 1/\omega$, $\chi(u) = u^{-1}$, $j = 2$, $k = 0$ and $m = 5$. From (6.8) and (6.11) we obtain

$$f^{\frac{1}{4}}(z)\, w(u,z) \underset{(2)}{=} -2\mathrm{i}\cos\left(\tfrac{1}{5}\pi\right)\{1+O(u^{-}\mathrm{e}^{u\xi_{1/\omega}(z)\frac{4}{5}})\} \quad (z \in [b_{2,1}, a_{2,1})_{\mathscr{P}}). \tag{9.3}$$

Because $k-j$ is negative, it also follows from theorem 6.1 that the branch of $f^{\frac{1}{4}}(z)$ in (9.3) is obtained from that used in (9.2) by continuous passage along $\mathscr{P}$, skirting $1/\omega$ in the clockwise sense. Had we enumerated the domains having $1/\omega$ on their boundary in a different manner, we could have taken $k-j = 3$. In place of (9.3) we would then obtain

$$f^{\frac{1}{4}}(z)\, w(u,z) \underset{(2)}{=} -2\cos\left(\tfrac{1}{5}\pi\right)\{1+O(u^{-\frac{4}{5}})\}\, \mathrm{e}^{u\xi_{1/\omega}(z)} \quad (z \in [b_{2,1}, a_{2,1})_{\mathscr{P}}), \tag{9.4}$$

where the continuation of $f^{\frac{1}{4}}(z)$ is now achieved by skirting $1/\omega$ in the anticlockwise sense. That (9.3) and (9.4) are consistent is easily seen from the fact that on completion of a simple closed circuit of $1/\omega$ in the positive sense the solution $w(u,z)$ is unaffected, but because $f(z)$ has a triple zero at $1/\omega$ the branch of $f^{\frac{1}{4}}(z)$ is changed by the factor $\mathrm{e}^{\frac{3}{2}\pi\mathrm{i}}$.

### 9.3. *Extension to domains*

Suppose now that instead of tracing the L.G. form of the solution $w(u,z)$, defined by (9.2), from one path segment to another, we seek asymptotic approximations of $w(u,z)$ for large $u$ that are uniform with respect to $z$ in the intersection of $\boldsymbol{D}_{2,1}$ with the annulus $\boldsymbol{A}_\omega$ defined by $|z-\omega| \geqslant r$, where $r$ is an arbitrary small positive constant. We first construct the L.G. approximation to $w(u,z)$ at the interior point $b_{2,0}$ of $\boldsymbol{D}_{2,0}$ by passage along $\mathscr{P}$ through $1/\omega$. The result is evidently given by (9.3) with $z = b_{2,0}$. To prepare for passage through $\omega$, we use the identity

$$\xi_{1/\omega}(z) = \xi_{1/\omega}(\omega) - \xi_\omega(z) \quad (z \in \boldsymbol{D}_{2,0} \cup \boldsymbol{D}_{3,0});$$

compare (7.4). We take a new progressive path with respect to $\omega$ that begins at $b_{2,0}$, passes through $\omega$ and continues in $\boldsymbol{D}_{2,1}$ along the curve having the equation $\operatorname{ph} \xi_\omega(z) = \theta$, where $\theta$ is a parameter in the closed interval $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$. (This path is not indicated in figure 13.) Theorem 6.2 is applied with $c = \omega$, $\chi(u) = u^{-\frac{4}{5}}$, $j = 0$, $k = 1$ and $m = 5$. Using (6.24), with lower signs, we obtain†

$$f^{\frac{1}{4}}(z)\, w(u,z) \underset{(2)}{=} -2\mathrm{i}\cos\left(\tfrac{1}{5}\pi\right)\{1 + O(u^{-\frac{4}{5}})\}\, \mathrm{e}^{u\xi_{1/\omega}(\omega) + u\xi_\omega(z)} \tag{9.5}$$

as $u \to \infty$, uniformly in the intersection of $\boldsymbol{A}_\omega$ and the right part of $\boldsymbol{D}_{2,1}$. Similarly from (6.24), with upper signs, we obtain

$$f^{\frac{1}{4}}(z)\, w(u,z) \underset{(2)}{=} -2\mathrm{i}\cos\left(\tfrac{1}{5}\pi\right)\{1 + O(u^{-\frac{4}{5}})\}\, \mathrm{e}^{u\xi_{1/\omega}(\omega) + u\xi_\omega(z)} + 4\cos^2\left(\tfrac{1}{5}\pi\right)\{1 + O(u^{-\frac{4}{5}})\}\, \mathrm{e}^{u\xi_{1/\omega}(\omega) - u\xi_\omega(z)} \tag{9.6}$$

as $u \to \infty$, uniformly in the intersection of $\boldsymbol{A}_\omega$ and the left part of $\boldsymbol{D}_{2,1}$.

In both (9.5) and (9.6) the branch of $f^{\frac{1}{4}}(z)$ is obtained from that used in (9.2) by continuous change from $(a_{3,2}, b_{3,2}]_{\mathscr{P}}$ to $\boldsymbol{D}_{2,1}$ passing the branch-points $1/\omega$ and 1 on the right, and $\omega$ on the left.

The regions of validity that we have quoted for (9.5) and (9.6) are not maximal. For example, we may pass from $b_{2,0}$ into $\boldsymbol{D}_{2,2}$ by using theorem 6.2 with $c = \omega$, $j = 0$, $k = 2$ and $m = 5$. We then find (9.6) is also uniformly valid in the intersection of $\boldsymbol{A}_\omega$ and the right part of $\boldsymbol{D}_{2,2}$, provided that $\xi_\omega(z)$ is replaced by $-\xi_\omega(z)$ at both occurrences (because $\xi_\omega(z)$ changes sign on crossing the boundary $\boldsymbol{D}_{2,1} \cap \boldsymbol{D}_{2,2}$).

## 10. Example 3: two fractional transition points and two simple poles

### 10.1. *Formulation of the problems*

The final example is furnished by the differential equation (8.1) with

$$f(z) = (z^2 + \beta^2)^{\frac{1}{2}}/(z^2 - \alpha^2), \tag{10.1}$$

$\alpha$ and $\beta$ being positive constants. Thus $f(z)$ has simple poles at $\pm\alpha$, and branch-points at $\pm \mathrm{i}\beta$. From our standpoint we regard $\pm\alpha$ as fractional transition points of order $-1$, and $\pm\mathrm{i}\beta$ as fractional transition points of order $\frac{1}{2}$. Taking the branch of $f(z)$ that is positive in the interval $(\alpha, \infty)$, we know from L.G. theory that there is a unique solution $w(u,z)$ of the differential equation with the property

$$w(u,z) \sim z^{\frac{1}{4}}\, \mathrm{e}^{-2uz^{\frac{1}{2}}} \quad (z \to +\infty), \tag{10.2}$$

the branches of $z^{\frac{1}{4}}$ and $z^{\frac{1}{2}}$ both being real and positive. The problems that we set ourselves are to find the asymptotic behaviour of $w(u,z)$ as $z \to -\infty$ in the following cases:

(*a*) $w(u,z)$ is continued from $z = +\infty$ along the positive real axis until the neighbourhood of $\alpha$ is reached. We then make $p + \frac{1}{2}$ circuits of $\alpha$, arriving at a point in the interval $(-\alpha, \alpha)$. We continue along the real axis until the neighbourhood of $-\alpha$ is reached and then make $q + \frac{1}{2}$ circuits of $-\alpha$, arriving at a point in the interval $(-\infty, -\alpha)$. Finally $w(u,z)$ is continued along the negative real axis until $-\infty$ is approached. The integers $p$ and $q$ may have any values, positive, zero or negative.

(*b*) This begins as in problem (*a*), but on reaching the origin we make a detour along the positive imaginary axis, encircle $\mathrm{i}\beta$ once in the positive sense and return to the origin along the positive imaginary axis. The rest of the journey, from 0 to $-\infty$, is completed as in (*a*).

(*c*) This begins as in problem (*b*), but on returning to the origin after passing around $\mathrm{i}\beta$ we continue along the negative imaginary axis, encircle $-\mathrm{i}\beta$ once in the positive sense and return to

† In (6.24) the term $\lambda_{j,k-1}$ disappears, and the error term $O(u^{-\frac{4}{5}})\, \mathrm{e}^{-u\xi_\omega(z)}$ is absorbable in the other error term.



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the origin along the negative imaginary axis. The rest of the journey, from 0 to $-\infty$, is completed as in (*a*).

These are not the only continuations from $+\infty$ to $-\infty$ that are possible around the four singularities, but they suffice for the purpose of illustration. It should be noted, however, that because all solutions of the differential equation are expansible in convergent series of ascending powers of $(z-\mathrm{i}\beta)^{\frac{1}{2}}$ in the neighbourhood of $\mathrm{i}\beta$, the effect of making any even number of circuits around this point is to leave $w(u,z)$ unchanged, and the effect of making any odd number of circuits is the same as making a single circuit. Similarly for circuits around $-\mathrm{i}\beta$.

### 10.2. *Topology of the principal regions*

The statements in the following three paragraphs are easily verified by means of conformal mapping on the plane of the function $\xi(z)$ defined by (7.3) and (10.1).

Five principal curves emerge from $\mathrm{i}\beta$. They occupy two Riemann sheets, and the phases of their initial directions are $\frac{1}{10}(8s-9)\,\pi$, $s = 1, 2, 3, 4, 5$. The first of these curves lies in the first and fourth quadrants, is symmetric with respect to the real axis and terminates at $-\mathrm{i}\beta$. The curve for $s = 2$ is confined to the second quadrant and tends to infinity asymptotically as a parabolic arc, with axis along the real axis. The curve for $s = 3$ is the segment of the imaginary axis from $\mathrm{i}\beta$ to $-\mathrm{i}\beta$. For $s = 4$ and 5 the curves are the reflections in the imaginary axis of those for $s = 2$ and 1, respectively.

Five principal curves emerge from $-\mathrm{i}\beta$. They are the complex conjugates of those emerging from $\mathrm{i}\beta$.

Associated with $\alpha$ there is an infinity of Riemann sheets, and only one principal curve lies on each. If we are using the branch of $f(z)$ for which $(z^2-\alpha^2)f(z)$ is positive on the real axis, then the principal curve is the interval $[-\alpha, \alpha]$. On the other hand, when $(z^2-\alpha^2)\,f(z)$ is negative on the real axis, the principal curve is the interval $[\alpha, \infty)$. Similarly for the transition point $-\alpha$.

The complete set of principal curves is indicated by the straight lines and curves sketched in figures 14, 15 and 16. Associated with the transition point $\mathrm{i}\beta$ there are five closed domains bounded by principal curves emanating from all four transition points. They are labelled $\boldsymbol{D}_{1,0}$, $\boldsymbol{D}_{1,1}$, $\boldsymbol{D}_{1,2}$, $\boldsymbol{D}_{1,3}$ and $\boldsymbol{D}_{1,4}$ on proceeding in the anticlockwise sense, and they occupy two Riemann sheets. When associated with $-\mathrm{i}\beta$ the same domains are relabelled $\boldsymbol{D}_{2,0}$, $\boldsymbol{D}_{2,4}$, $\boldsymbol{D}_{2,3}$, $\boldsymbol{D}_{2,2}$, $\boldsymbol{D}_{2,1}$, as indicated in the diagrams.

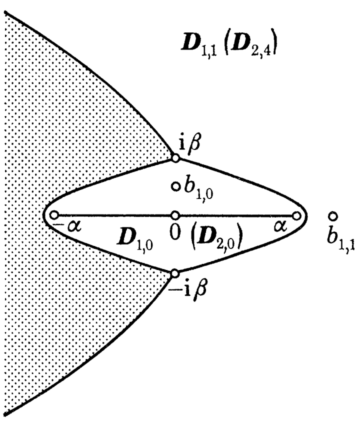


FIGURE 14. Example 3: *z*-plane (i).

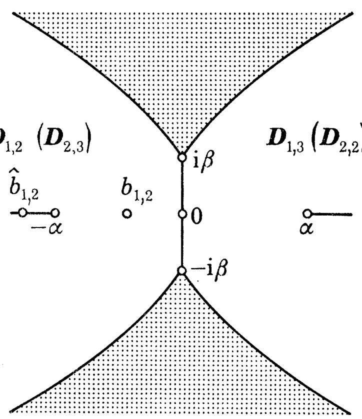


FIGURE 15. Example 3: *z*-plane (ii).

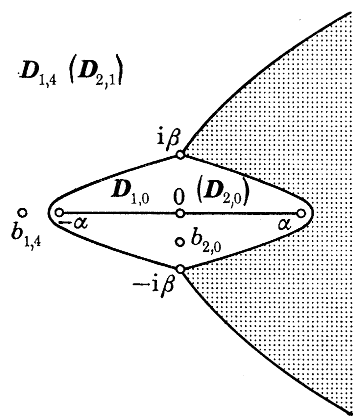


FIGURE 16. Example 3: *z*-plane (iii).

The boundaries of $\boldsymbol{D}_{1,0}$ (or $\boldsymbol{D}_{2,0}$) are the two principal curves that join $\mathrm{i}\beta$ and $-\mathrm{i}\beta$, and the interval $[-\alpha, \alpha]$. Thus $\boldsymbol{D}_{1,0}$ is multiply connected, and continuous passage around any closed contour in $\boldsymbol{D}_{1,0}$ that contains the interval $[-\alpha, \alpha]$ in its interior alters the value of each solution of the differential equation and also the value of the function $\xi(z)$, defined by (7.3) and (10.1). If we permit any number of such circuits, then the map of $\boldsymbol{D}_{1,0}$ on the $\xi$-plane is a vertical strip. The other domains are simply connected. The $\xi$-maps of $\boldsymbol{D}_{1,1}$ and $\boldsymbol{D}_{1,4}$ are half-planes, and those of $\boldsymbol{D}_{1,2}$ and $\boldsymbol{D}_{1,3}$ are vertical strips.

### 10.3. *Problem* (*a*)

Let $b_{1,1}$ and $b_{1,4}$ be arbitrary fixed points on the real axis such that $b_{1,1} > \alpha$ and $b_{1,4} < -\alpha$; see figures 14 and 16. From L.G. theory there is a solution $\hat{w}(u,z)$ of the differential equation with the properties

$$f^{\frac{1}{4}}(z)\,\hat{w}(u,z) \sim \mathrm{e}^{-u\xi_\alpha(z)} \tag{10.3}$$

as $z \to +\infty$, and

$$f^{\frac{1}{4}}(z)\,\hat{w}(u,z) \underset{(2)}{=} \{1+O(u^{-1})\}\,\mathrm{e}^{-u\xi_\alpha(z)} \quad (z \in [b_{1,1}, \infty)) \tag{10.4}$$

as $u \to \infty$. The branch of $f^{\frac{1}{4}}(z)$ is taken to be real and positive in both relations. From (10.1) and the definition of $\xi_\alpha(z)$, given by (7.1) with $c = \alpha$, we see that

$$\xi_\alpha(z) = 2z^{\frac{1}{2}} + \kappa + O(z^{-\frac{3}{2}}) \quad (z \to +\infty),$$

where $\kappa$ is the real constant defined by

$$\kappa = \int_\alpha^\infty \left\{\frac{(t^2+\beta^2)^{\frac{1}{4}}}{(t^2-\alpha^2)^{\frac{1}{2}}} - \frac{1}{t^{\frac{1}{2}}}\right\} \mathrm{d}t - 2\alpha^{\frac{1}{2}}. \tag{10.5}$$

On comparing (10.2) and (10.3), we see that the relation of $\hat{w}(u,z)$ to the wanted solution $w(u,z)$ is given by

$$w(u,z) = \mathrm{e}^{\kappa u}\hat{w}(u,z). \tag{10.6}$$

Let us continue $\hat{w}(u,z)$ from $[b_{1,1}, \infty)$ to the point $0+0\mathrm{i}$ on the upper side of the principal curve joining $\alpha$ and $-\alpha$, making $p$ circuits of $\alpha$ *en route* but not encircling any of the other transition points. In applying theorem 6.1 we take the progressive path $\mathscr{P}$ along the positive real axis, $c = \alpha$, $\chi(u) = u^{-1}$, $k-j = p$ and $m = 1$. Because $0+0\mathrm{i}$ is a left boundary point when viewed from $\alpha$, the appropriate formulae are (6.9), (6.11) and (6.12). Referring to (2.18) and (4.8), we find that

$$f^{\frac{1}{4}}(z)\,\hat{w}(u,z) \underset{(2)}{=} \mathrm{i}^{1-p}p\{1+O(u^{-1})\}\,\mathrm{e}^{u\xi_\alpha(z)} + \mathrm{i}^{-p}(p+1)\{1+O(u^{-1})\}\,\mathrm{e}^{-u\xi_\alpha(z)} \quad (z = 0+0\mathrm{i}). \tag{10.7}$$

To prepare for passage around $-\alpha$ we observe that if $z$ tends to $-\alpha$ along the upper side of the interval $[-\alpha, \alpha]$, then $\xi_\alpha(z)$ tends to $2\mathrm{i}\rho$ where $\rho$ is the positive real constant defined by

$$\rho = \int_0^\alpha |f^{\frac{1}{2}}(t)\,\mathrm{d}t| = \int_0^\alpha \frac{|\beta^2+t^2|^{\frac{1}{4}}}{|\alpha^2-t^2|^{\frac{1}{2}}}\,\mathrm{d}t; \tag{10.8}$$

compare § 5.1. We therefore recast (10.7) in the form

$$f^{\frac{1}{4}}(z)\,\hat{w}(u,z) \underset{(2)}{=} \mathrm{i}^{1-p}p\,\mathrm{e}^{2\mathrm{i}\rho u}\{1+O(u^{-1})\}\,\mathrm{e}^{u\xi_{-\alpha}(z)} + \mathrm{i}^{-p}(p+1)\,\mathrm{e}^{-2\mathrm{i}\rho u}\{1+O(u^{-1})\}\,\mathrm{e}^{-u\xi_{-\alpha}(z)}, \tag{10.9}$$

again with $z = 0+0\mathrm{i}$; compare (7.5). After completing $q+\frac{1}{2}$ circuits of $-\alpha$ and arriving in the interval $(-\infty, b_{1,4}]$ (figure 16), we calculate the contribution from the second term on the right hand side of (10.9) by applying theorem 6.1 with $c = -\alpha$, $\chi(u) = u^{-1}$, $k-j = q$ and $m = 1$. This yields

$$\mathrm{i}^{-p}(p+1)\,\mathrm{e}^{-2\mathrm{i}\rho u}\,\mathrm{i}^{1-q}q\{1+O(u^{-1})\}\,\mathrm{e}^{u\xi_{-\alpha}(z)}. \tag{10.10}$$

To handle the contribution of the first term on the right hand side of (10.9), we proceed as in §7, case II and cross the principal curve joining $\alpha$ and $-\alpha$, passing from $0+0\mathrm{i}$ to $0-0\mathrm{i}$. This replaces $\xi_{-\alpha}(z)$ by $-\xi_{-\alpha}(z)$, and also takes us into another principal region associated with $-\alpha$. Accordingly, theorem 6.1 is now applied with $j=-1$, $k=q$ and $m=1$. This yields

$$\mathrm{i}^{1-p}p\,\mathrm{e}^{2\mathrm{i}\rho u}\,\mathrm{i}^{-q}(q+1)\{1+O(u^{-1})\}\,\mathrm{e}^{u\xi_{-\alpha}(z)}. \tag{10.11}$$

Combining (10.10) and (10.11) we obtain

$$f^{\frac{1}{4}}(z)\,\hat{w}(u,z) \underset{(2)}{=} \mathrm{i}^{1-p-q}\{(p+1)q\,\mathrm{e}^{-2\mathrm{i}\rho u}+p(q+1)\,\mathrm{e}^{2\mathrm{i}\rho u}+O(u^{-1})\}\,\mathrm{e}^{u\xi_{-\alpha}(z)} \quad (z\in(-\infty, b_{1,4}]). \tag{10.12}$$

The theory of the L.G. approximation shows that when the positive branch of $f(z)$ is used in the interval $(-\infty, -\alpha)$, the wanted solution $w(u,z)$ is expressible in the form

$$w(u,z) = A(u)\,|z|^{\frac{1}{4}}\,\mathrm{e}^{2u|z|^{\frac{1}{2}}}\{1+O(z^{-\frac{1}{2}})\}+B(u)\,|z|^{\frac{1}{4}}\,\mathrm{e}^{-2u|z|^{\frac{1}{2}}}\{1+O(z^{-\frac{1}{2}})\} \tag{10.13}$$

as $z\to-\infty$, where $A(u)$ and $B(u)$ are independent of $z$. The coefficient $B(u)$ of the recessive component cannot be found by the present analysis, but $A(u)$ can be estimated for large $u$ as follows. From the definition (7.1) of $\xi_{-\alpha}(z)$, we calculate that

$$\xi_{-\alpha}(z) = 2|z|^{\frac{1}{2}}+\kappa+O(z^{-\frac{3}{2}}) \quad (z\to-\infty), \tag{10.14}$$

where $\kappa$ is defined by (10.5). Next, because $f(z)$ has simple poles at $\pm\alpha$, the branch of $f^{\frac{1}{4}}(z)$ used in (10.12) is given by

$$\mathrm{e}^{-\frac{1}{4}(2p+1)\pi\mathrm{i}-\frac{1}{4}(2q+1)\pi\mathrm{i}}|z^2+\beta^2|^{\frac{1}{8}}|z^2-\alpha^2|^{-\frac{1}{4}}, \tag{10.15}$$

and is therefore asymptotic to $\mathrm{i}^{-p-q-1}|z|^{-\frac{1}{4}}$ as $z\to-\infty$. Substituting these estimates in (10.12) and comparing the result with (10.6) and (10.13), we see that

$$A(u) = -\mathrm{e}^{2\kappa u}\{(p+1)\,q\,\mathrm{e}^{-2\mathrm{i}\rho u}+p(q+1)\,\mathrm{e}^{2\mathrm{i}\rho u}+O(u^{-1})\}; \tag{10.16}$$

compare theorem 3.1 of Olver (1974, ch. 6). This is the required result.

### 10.4. *Problem* (*b*)

We follow the analysis of §10.3 until we reach (10.6). Instead of continuing $\hat{w}(u,z)$ from $[b_{1,1},\infty)$ to $0+0\mathrm{i}$, we continue this solution to a point $b_{1,0}$ in $\boldsymbol{D}_{1,0}$ on the join of 0 and $\mathrm{i}\beta$; see figure 14. By applying theorem 6.1, using (6.8) and (6.11), we obtain

$$f^{\frac{1}{4}}(z)\,\hat{w}(u,z) \underset{(2)}{=} \mathrm{i}^{1-p}p\{1+O(u^{-1})\}\,\mathrm{e}^{u\xi_{\alpha}(z)} \quad (z=b_{1,0}); \tag{10.17}$$

compare (10.7).

To prepare for passage around $\mathrm{i}\beta$, we note that when $\mathrm{i}\beta$ is regarded as a member of $\boldsymbol{D}_{1,0}$ we have $\xi_\alpha(\mathrm{i}\beta)=\sigma+\mathrm{i}\rho$, where $\rho$ is defined by (10.8) and $\sigma$ is the positive real constant given by

$$\sigma = \int_0^{\mathrm{i}\beta}|f^{\frac{1}{2}}(t)\,\mathrm{d}t| = \int_0^{\beta}\frac{|\beta^2-t^2|^{\frac{1}{4}}}{|\alpha^2+t^2|^{\frac{1}{2}}}\,\mathrm{d}t. \tag{10.18}$$

Equation (10.17) is therefore recast in the form

$$f^{\frac{1}{4}}(z)\,\hat{w}(u,z) \underset{(2)}{=} \mathrm{i}^{1-p}p\,\mathrm{e}^{(\sigma+\mathrm{i}\rho)u}\{1+O(u^{-1})\}\,\mathrm{e}^{-u\xi_{\mathrm{i}\beta}(z)} \quad (z=b_{1,0}); \tag{10.19}$$

compare (7.4). We now continue $\hat{w}(u,z)$ to an interior point $b_{1,2}$ of $\boldsymbol{D}_{1,2}$; see figure 15. This continuation is achieved by applying theorem 6.1 with $c=\mathrm{i}\beta$, $\chi(u)=u^{-1}$, $j=0$, $k=2$ and $m=\frac{5}{2}$. From (6.8) and (6.11) we obtain

$$f^{\frac{1}{4}}(z)\,\hat{w}(u,z) \underset{(2)}{=} 2\cos\left(\tfrac{2}{5}\pi\right)\,\mathrm{i}^{2-p}p\,\mathrm{e}^{(\sigma+\mathrm{i}\rho)u}\{1+O(u^{-1})\}\,\mathrm{e}^{u\xi_{\mathrm{i}\beta}(z)} \quad (z=b_{1,2}).$$



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To prepare for passage around $-\alpha$ we note that when $-\alpha$ is regarded as a member of $\boldsymbol{D}_{1,2}$ we have $\xi_{i\beta}(-\alpha) = \rho+i\sigma$, where $\rho$ and $\sigma$ are defined by (10.8) and (10.18). Accordingly, we recast the last equation in the form

$$f^{\frac14}(z)\,\hat{w}(u,z) \underset{(2)}{=} 2\cos(\tfrac25\pi)\, i^{2-p}p\, e^{(\rho+\sigma)(1+i)u}\{1+O(u^{-1})\}\, e^{-u\xi_{-\alpha}(z)} \quad (z=b_{1,2});$$

compare (7.4). Theorem 6.1 is now applied with $c = -\alpha$, $\chi(u) = u^{-1}$, $k-j = q$ and $m = 1$. From (6.9), (6.11) and (6.12), we obtain

$$f^{\frac14}(z)\,\hat{w}(u,z) \underset{(2)}{=} 2\cos(\tfrac25\pi)\, i^{2-p}p\, e^{(\rho+\sigma)(1+i)u}[i^{1-q}q\{1+O(u^{-1})\}\, e^{u\xi_{-\alpha}(z)} + i^{-q}(q+1)\{1+O(u^{-1})\}\, e^{-u\xi_{-\alpha}(z)}] \quad (z\in(-\infty,\hat{b}_{1,2}]), \tag{10.20}$$

where $\hat{b}_{1,2}$ is an arbitrary fixed point of the interval $(-\infty, -\alpha)$; see figure 15.

The theory of the L.G. approximation shows that when the negative branch of $f(z)$ is used in the interval $(-\infty, -\alpha)$ the wanted solution $w(u,z)$ is expressible in the form

$$w(u,z) = A_1(u)\,|z|^{\frac14}\, e^{2iu|z|^{\frac12}}\{1+O(z^{-\frac32})\} + A_2(u)\,|z|^{\frac14}\, e^{-2iu|z|^{\frac12}}\{1+O(z^{-\frac32})\} \tag{10.21}$$

as $z\to-\infty$, where $A_1(u)$ and $A_2(u)$ are independent of $z$. In the present circumstances we have

$$\xi_{-\alpha}(z) = 2i|z|^{\frac12} + i\kappa + O(z^{-\frac32}) \quad (z\to-\infty);$$

compare (10.14). Also, the appropriate branch of $f^{\frac14}(z)$ is $e^{\frac14\pi i}$ times the expression (10.15); accordingly

$$f^{\frac14}(z) \sim e^{-\frac14\pi i}\, i^{-p-q}|z|^{-\frac14} \quad (z\to-\infty).$$

On substituting these estimates in (10.20) and comparing the result with (10.6) and (10.21), we conclude that

$$\left.\begin{aligned} A_1(u) &= 2\cos(\tfrac25\pi)\, pq\, e^{-\frac14\pi i}\, e^{(\rho+\sigma+\kappa)(1+i)u}\{1+O(u^{-1})\},\\ A_2(u) &= -2\cos(\tfrac25\pi)\, p(q+1)\, e^{\frac14\pi i}\, e^{(\rho+\sigma)(1+i)u}\, e^{\kappa(1-i)u}\{1+O(u^{-1})\}.\end{aligned}\right\} \tag{10.22}$$

This is the required result.

### 10.5. *Problem* (*c*)

The analysis in this case is similar to that of §§10.3 and 10.4 and it suffices to sketch the main steps and state the result.

After reaching (10.19) we continue to $z = 0$ ($\in \boldsymbol{D}_{1,2}$) by use of (6.9), (6.11) and (6.12). We then pass around $-i\beta$ by the method of §7, case II, arriving at an arbitrary point $b_{2,0}$ on the join of 0 and $-i\beta$ in $\boldsymbol{D}_{2,0}$; see figure 16. Lastly, we pass around $-\alpha$ by the method of §7, case I, arriving at $(-\infty, b_{1,4}]$. The final result is expressed by (10.13), with

$$A(u) = -2\cos(\tfrac25\pi)\sec(\tfrac15\pi)\, p(q+1)\, e^{2(\kappa+\sigma+i\rho)u}\{\cos(2\sigma u)+O(u^{-1})\}, \tag{10.23}$$

and $B(u)$ again undetermined.

## Appendix. Computation of principal curves

The principal curves in figures 12, 13, 14, 15 and 16 were computed on the UNIVAC 1108 computer and plotted on the TEKTRONIX 4013 plotter at the University of Maryland, using the following method.

Let $c$ again denote the transition point under consideration and $m-2$ its multiplicity. Then

$$f(z) = (z-c)^{m-2}\tilde{f}(z), \tag{A 1}$$


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where $\tilde{f}(z)$ is analytic and non-vanishing at $c$. Each principal curve emerging from $c$ has the equation

$$\mathrm{Re}\int_c^z f^{\frac{1}{2}}(t)\,\mathrm{d}t = 0. \tag{A 2}$$

These curves are computed in parametric form

$$z = x(\tau) + \mathrm{i}y(\tau), \tag{A 3}$$

where $\tau$ is the arc parameter measured from $c$, and $x(\tau)$ and $y(\tau)$ denote the real and imaginary parts of $z$, respectively.

Differentiation yields

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\int_c^z f^{\frac{1}{2}}(t)\,\mathrm{d}t = (z-c)^{\frac{1}{2}(m-2)}\tilde{f}^{\frac{1}{2}}(z)\left(\frac{\mathrm{d}x}{\mathrm{d}\tau}+\mathrm{i}\frac{\mathrm{d}y}{\mathrm{d}\tau}\right). \tag{A 4}$$

Let $\theta(\tau)$ denote the inclination of the curve (A 3) to the real axis, so that

$$\mathrm{d}x/\mathrm{d}\tau = \cos\theta, \quad \mathrm{d}y/\mathrm{d}\tau = \sin\theta. \tag{A 5}$$

The condition that the real part of the right hand side of (A 4) be zero yields

$$\theta = -\tfrac{1}{2}(m-2)\arctan\left(\frac{y-c_{\mathrm{I}}}{x-c_{\mathrm{R}}}\right) - \tfrac{1}{2}\arctan\left\{\frac{\tilde{f}_{\mathrm{I}}(x+\mathrm{i}y)}{\tilde{f}_{\mathrm{R}}(x+\mathrm{i}y)}\right\} + (mp+q+\tfrac{1}{2})\pi, \tag{A 6}$$

where $c_{\mathrm{R}}$, $c_{\mathrm{I}}$, $\tilde{f}_{\mathrm{R}}(x+\mathrm{i}y)$ and $\tilde{f}_{\mathrm{I}}(x+\mathrm{i}y)$ denote the real part of $c$, the imaginary part of $c$, the real part of $\tilde{f}(z)$ and the imaginary part of $\tilde{f}(z)$, respectively. Also, $p$ and $q$ are arbitrary integers, and it is assumed that the inverse tangents occupy the same quadrants as the points $(x-c_{\mathrm{R}})+\mathrm{i}(y-c_{\mathrm{I}})$ and $\tilde{f}_{\mathrm{R}}(x+\mathrm{i}y)+\mathrm{i}\tilde{f}_{\mathrm{I}}(x+\mathrm{i}y)$, respectively.

The principal curves are computed by numerical integration of the simultaneous nonlinear differential equations represented by (A 5) and (A 6), using the initial conditions

$$x(0) = c_{\mathrm{R}}, \quad y(0) = c_{\mathrm{I}}, \quad \theta(0) = \{(2s+1)\pi - \mathrm{ph}\tilde{f}(c)\}/m,$$

where $s$ is an integer that corresponds to the particular curve under consideration.

The integers $p$ and $q$ in (A 6) are determined by continuity. Thus if $\tau_{n-1}$ and $\tau_n$ are successive discrete values of $\tau$, then $|\theta(\tau_n)-\theta(\tau_{n-1})|$ is required to lie within a prescribed tolerance. For each of the examples treated in part C a suitable value for this tolerance was found to be $\frac{1}{4}\pi$.

No problems of stiffness arise in the integration of equations (A 5) and (A 6); in consequence any appropriate initial-value method may be employed. The program actually used was based on the codes DE, STEP and INTRP described in ch. 10 of Shampine & Gordon (1975). Each of the principal curves in examples 1, 2 and 3 of part C was computed for a total arc length of 12 units (the arc lengths in the diagrams are somewhat less), the average computing time being just under 4 seconds.

*Remark*. In place of (A 6), we could employ the simpler formula

$$\theta = -\tfrac{1}{2}\arctan\{f_{\mathrm{I}}(x+\mathrm{i}y)/f_{\mathrm{R}}(x+\mathrm{i}y)\} + (q+\tfrac{1}{2})\pi, \tag{A 7}$$

where $f_{\mathrm{R}}(x+\mathrm{i}y)$ and $f_{\mathrm{I}}(x+\mathrm{i}y)$ denote the real and imaginary parts of $f(z)$, respectively. The disadvantage of (A 7) compared with (A 6) is that $f_{\mathrm{R}}(x+\mathrm{i}y)$ and $f_{\mathrm{I}}(x+\mathrm{i}y)$ often underflow when $\tau$ is small, owing to the presence of the factor $(z-c)^{m-2}$ on the right hand side of (A 1).

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